2. <u>The Axiomatic Translation of Modal Logic</u> [1].

This entire section is based upon, but heavily modified from, the text of [1]. The material has been re-ordered to better follow a narrative description, and many derivations and examples have been added throughout, to support this description. For formal definitions of the formulae presented see [1]. Some material is extended from results suggested in [1].

2.1 Axiomatic translation of the input modal problem.

If the complications are ignored, then the axiomatic translation is quite simple. A modal formula φ is processed (under the function $\prod$) as seen in formula 2.1. All modal (that is, box) symbols are eliminated by this translation, giving rise to a formula in first-order logic.

$$\prod(\varphi) = \exists x\, Q_\varphi(x) \wedge \bigwedge\{Def(\psi) \mid \psi \in Sf(\varphi)\}$$

formula 2.1 (see definition 4.1 in [1])

In formula 2.1 the first component is existentially quantified, and the translation is then for local satisfiability. (For global satisfiability the quantification is universal).

In formula 2.1 $Sf(\varphi)$ denotes the set of all *sub-formulae* of φ, and Q_φ is a predicate symbol uniquely associated with the modal formula φ. The sub-formulae are each processed as shown in the definition of $Def(\psi)$ (see formulae 2.2 and 2.3), and the results are added to others by repeated conjugation ($\wedge$).

$$\begin{aligned} Def(\psi) \quad &= \quad \forall x(Q_\psi(x) \rightarrow \pi(\psi, x)) \\ &\wedge \quad \forall x(Q_\psi(x) \leftrightarrow \neg Q_{\sim\psi}(x)) \\ &\wedge \quad \forall x(Q_{\sim\psi}(x) \rightarrow \pi(\sim\psi, x)) \end{aligned}$$

[from $\forall x(Q_\psi(x)\rightarrow\neg Q_{\sim\psi}(x))\wedge\forall x(Q_\psi(x)\leftarrow\neg Q_{\sim\psi}(x))$]

[formula 2.2, see section 3 in [1]]

This definition includes the (optional) reverse implication (see section 2.2.4)

Complement (~) is defined so $\sim\phi=\neg\phi$ and $\sim\neg\phi=\phi$.

where $\pi(\bot, x) = \bot$

$$\pi(p, x) = \top \qquad \pi(\neg p, x) = \neg Q_p(x)$$

$$\pi(\psi\wedge\phi, x) = Q_\psi(x) \wedge Q_\phi(x) \qquad \pi(\neg(\psi\wedge\phi), x) = Q_{\sim\psi}(x) \vee Q_{\sim\phi}(x)$$

$$\pi(\Box\psi, x) = \forall y(R(x, y) \rightarrow Q_\psi(y))$$

$$\pi(\neg\Box\psi, x) = \exists y(R(x, y) \wedge Q_{\sim\psi}(y))$$

[formulae 2.3, see section 3 in [1]]

$$\pi(\psi\wedge\phi\wedge\varphi, x) = Q_\psi(x) \wedge Q_\phi(x) \wedge Q_\varphi(x)$$

$$\pi(\neg(\psi\wedge\phi\wedge\varphi), x) = Q_{\sim\psi}(x) \vee Q_{\sim\phi}(x) \vee Q_{\sim\varphi}(x)$$

[formulae 2.3.1]

p is a predicate; ψ,ϕ,φ are modal formulae;

x and y are distinct free variables; R is the accessibility relation

An obvious extension is to consider the non-binary case of conjunction, for example in the ternary case, as seen in formula 2.3.1. It is easy to show that under resolution, the clauses for example from Def(ψ∧ϕ∧φ) are identical in nature to those arising from the binary cases Def((ψ∧ϕ)∧φ), Def(ψ∧(ϕ∧φ)), and Def(ϕ∧(φ∧ψ)). A similar result can be inferred for any number of conjugated formulae.

Note that the translation of the modal formula (φ) has a single iteration, unlike the classical translation that applies the formulae 1.4 inductively (in multiple iterations) until all modal symbols are eliminated. Here the iteration is replaced by consideration of all of the sub-formulae of φ. The resulting definitions are linked, one to another, by the predicate symbols that are generated (each uniquely associated with a particular sub-formulae of the target modal formula). It should be noted that the translation is only defined for formulae with the operators: ∧, ¬, and □ (and with ⊥). Any modal formula to be translated must first be modified, by simple transformations (see section 5.3), to exclude other operators. This simplifies the axiomatic translation. An example of *manual* manipulation of a modal formula under axiomatic translation is shown in figure 2.2, taking as an example the modal formula φ = □((¬□(r,p)) ∧ □(r,q)). First, it is necessary to define the sub-formulae of the modal formula φ, as marked by an arrow in figure 2.1. Essentially every possible sub-formula that can be defined within the parent formula is identified. This set of sub-formulae is known as the default *instantiation set* for this modal formula. Note, that for simplicity the modality index, r has been dropped in the any example where it is possible (that is, where the example is uni-modal). In all examples, R is the binary predicate symbol associated with the modality index r.

Considering the translation in figure 2.2, it is clear that while the transformations in the axiomatic translation are simple, they give rise to a *great many* new sub-formulae. (It can be shown [1] that the instantiation set and the translation and with the extensions for incorporation of modal axioms described in later sections, can be calculated in linear time). Fortunately, there are some further modifications, as illustrated in figures 2.3-2.5, that lead to a considerable simplification. First, if p is a predicate, then Def(p) can be simplified as shown in formula 2.4. The derivation is seen in figure 2.3

Where ψ is a predicate formula 2.2 can be simplified to

$$\mathrm{Def}(\psi) \;=\; \forall x(Q_{\psi}(x) \leftrightarrow \neg Q_{\sim\psi}(x))$$

[from $\forall x(Q_{\psi}(x)\rightarrow \neg Q_{\sim\psi}(x))\wedge\forall x(Q_{\psi}(x)\leftarrow \neg Q_{\sim\psi}(x))$]
[formula 2.4]

This definition includes the (optional) reverse implication (see section 2.2.4).

Second, if a modal formula with an enclosing ‘not’ symbol (¬) is considered, then it can be shown that Def(¬ψ) andDef(ψ) are equivalent Since the sub-formulae of ¬ψmust include ψ, inclusion of sub-formulae with an enclosing ¬ symbol in the instantiation set is redundant. The proof (shown in figure 2.4) involves comparison of non-negated and negated standardized modal formulae, considering all three cases that might occur. These optimizations indicate that Def(ψ) formulae involving negated symbols ($\neg p$, $\neg\Box p$, or $\neg(p\wedge q)$) can be omitted from the axiomatic translation (shown in figures 2.1 and 2.2). Using just these two modifications it is possible to reduce the translation in figure 2.2 to that shown in figure 2.5. It is clear that the number of terns submitted to resolution will be greatly reduced, and this is likely to yield a faster and shorter proof.

Although it is not relevant to the previous example, a further obvious modification arises from the translation of conjuncted formulae. The new predicate symbols introduced are different, if for example, $Q_{\psi\wedge\phi}$ and $Q_{\phi\wedge\psi}$ are considered. Clearly these predicates should be the same under translation. Without optimization duplication of formulae is likely to occur, that will increase the size of the task presented to the SPASS resolution prover (for example in the target formula $(\psi\wedge\phi) \rightarrow (\phi\wedge\psi)$). The solution is to sort the arguments in the list of an $\wedge$ operator. The sorting function is unimportant, as long as it always returns the same result for a given series of formulae.

Figure 2.1 <u>Definition of the instantiation set</u>.

The full instantiation set of the target formula $\varphi = \Box((\neg\Box(r,p)) \wedge \Box(r,q))$ is shown.

↓		↓	↓	↓	↓	↓	↓	
$\Box$	(	$(\neg$	$\Box$	p)	$\wedge$	$\Box$	q	)

Listing the sub-formulae gives

1.	$\Box((\neg\Box p)\wedge\Box q)$	5.	p
2.	$\neg\Box p\wedge\Box q$	6.	$\Box q$
3.	$\neg\Box p$	7.	q
4.	$\Box p$		

Figure 2.2 <u>An example of an axiomatic translation</u>.

The axiomatic translation is given for the target formula $\varphi = \Box((\neg\Box(r,p)) \wedge \Box(r,q))$.

$\Pi(\Box((\neg\Box p)\wedge\Box q))$	$= \exists x Q_{\Box(\neg\Box p\wedge\Box q)}(x)$	
$\wedge$	$\forall x(Q_{\Box(\neg\Box p\wedge\Box q)}(x) \rightarrow \pi(\Box((\neg\Box p)\wedge\Box q), x))$	[from Def($\Box((\neg\Box p)\wedge\Box q)$) (Sf1)]
$\wedge$	$\forall x(Q_{\Box(\neg\Box p\wedge\Box q)}(x) \leftrightarrow \neg Q_{\sim\Box(\neg\Box p\wedge\Box q)}(x))$	
$\wedge$	$\forall x(Q_{\sim\Box(\neg\Box p\wedge\Box q)}(x) \rightarrow \pi(\sim\Box((\neg\Box p)\wedge\Box q), x))$	
$\wedge$	$\forall x(Q_{\neg\Box p\wedge\Box q}(x) \rightarrow \pi((\neg\Box p)\wedge\Box q, x))$	[from Def($(\neg\Box p)\wedge\Box q$) (Sf2)]
$\wedge$	$\forall x(Q_{\neg\Box p\wedge\Box q}(x) \leftrightarrow \neg Q_{\sim(\neg\Box p\wedge\Box q)}(x))$	
$\wedge$	$\forall x(Q_{\sim(\neg\Box p\wedge\Box q)}(x) \rightarrow \pi(\sim((\neg\Box p)\wedge\Box q), x))$	
$\wedge$	$\forall x(Q_{\neg\Box p}(x) \rightarrow \pi(\neg\Box p, x))$	[from Def($\neg\Box p$) (Sf3)]
$\wedge$	$\forall x(Q_{\neg\Box p}(x) \leftrightarrow \neg Q_{\sim\neg\Box p}(x))$	
$\wedge$	$\forall x(Q_{\sim\neg\Box p}(x) \rightarrow \pi(\sim\neg\Box p, x))$	
$\wedge$	$\forall x(Q_{\Box p}(x) \rightarrow \pi(\Box p, x))$	[from Def($\Box p$) (Sf4)]
$\wedge$	$\forall x(Q_{\Box p}(x) \leftrightarrow \neg Q_{\sim\Box p}(x))$	

$\wedge$	$\forall x(Q_{\sim\Box p}(x) \rightarrow \pi(\sim\Box p, x))$	
$\wedge$	$\forall x(Q_{\Box q}(x) \rightarrow \pi(\Box q, x))$	[from Def(□q) (Sf6)]
$\wedge$	$\forall x(Q_{\Box q}(x) \leftrightarrow \neg Q_{\sim\Box q}(x))$	
$\wedge$	$\forall x(Q_{\sim\Box q}(x) \rightarrow \pi(\sim\Box q, x))$	
$\wedge$	$\forall x(Q_{p}(x) \rightarrow \pi(p, x))$	[from Def(p) (Sf5)]
$\wedge$	$\forall x(Q_{p}(x) \leftrightarrow \neg Q_{\sim p}(x))$	
$\wedge$	$\forall x(Q_{\sim p}(x) \rightarrow \pi(\sim p, x))$	
$\wedge$	$\forall x(Q_{q}(x) \rightarrow \pi(q, x))$	[from Def(q) (Sf7)]
$\wedge$	$\forall x(Q_{q}(x) \leftrightarrow \neg Q_{\sim q}(x))$	
$\wedge$	$\forall x(Q_{\sim q}(x) \rightarrow \pi(\sim q, x))$	
$\prod(\Box((\neg\Box p) \wedge \Box q)) \quad =$	$\exists x Q_{\Box(\neg\Box p\wedge\Box q)}(x)$	
$\wedge$	$\forall x(Q_{\Box(\neg\Box p\wedge\Box q)}(x) \rightarrow \forall y(R(x,y)\rightarrow Q_{\neg\Box p\wedge\Box q}(y))$	[from Sf1)]
$\wedge$	$\forall x(Q_{\Box(\neg\Box p\wedge\Box q)}(x) \leftrightarrow \neg Q_{\neg\Box(\neg\Box p\wedge\Box q)}(x))$	$[Q_{\sim\Box(\neg\Box p\wedge\Box q)}(x)= Q_{\neg\Box(\neg\Box p\wedge\Box q)}(x)]$
$\wedge$	$\forall x(Q_{\neg\Box(\neg\Box p\wedge\Box q)}(x) \rightarrow \exists y(R(x,y)\wedge Q_{\neg(\neg\Box p\wedge\Box q)}(y))$	$[\sim\Box(\neg\Box p\wedge\Box q)=\neg\Box(\neg\Box p\wedge\Box q)]$
$\wedge$	$\forall x(Q_{\neg\Box p\wedge\Box q}(x) \rightarrow (Q_{\neg\Box p}(x)\wedge Q_{\Box q}(x))$	[from Sf2]
$\wedge$	$\forall x(Q_{\neg\Box p\wedge\Box q}(x) \leftrightarrow \neg Q_{\neg(\neg\Box p\wedge\Box q)}(x))$	
$\wedge$	$\forall x(Q_{\neg(\neg\Box p\wedge\Box q)}(x) \rightarrow (Q_{\Box p}(x)\vee Q_{\neg\Box q}(x))$	$[\sim(\neg\Box p\wedge\Box q)=\neg(\neg\Box p\wedge\Box q);$ $Q_{\sim(\neg\Box p\wedge\Box q)}(x)= Q_{\neg(\neg\Box p\wedge\Box q)}(x);$ $Q_{\neg\neg\Box p}(x)= Q_{\Box p}(x)]$
$\wedge$	$\forall x(Q_{\neg\Box p}(x) \rightarrow \exists y(R(x,y)\wedge Q_{\neg p}(y))$	[from Sf3]
$\wedge$	$\forall x(Q_{\neg\Box p}(x) \leftrightarrow \neg Q_{\Box p}(x))$	$[\sim\neg\Box p = \neg\neg\Box p = \Box p]$
$\wedge$	$\forall x(Q_{\Box p}(x) \rightarrow \forall y(R(x,y)\rightarrow Q_{p}(y))$	$[\sim\neg\Box p = \neg\neg\Box p = \Box p]$
$\wedge$	$\forall x(Q_{\Box p}(x) \rightarrow \forall y(R(x,y)\rightarrow Q_{p}(y))$	[from Sf4]
$\wedge$	$\forall x(Q_{\Box p}(x) \leftrightarrow \neg Q_{\neg\Box p}(x))$	$[Q_{\sim\Box}p = Q_{\neg\Box}p]$
$\wedge$	$\forall x(Q_{\neg\Box p}(x) \rightarrow \exists y(R(x,y)\wedge Q_{\neg p}(y))$	$[Q_{\sim p}(x)= Q_{\neg p}(x);\ Q_{\sim\Box p}(x) = Q_{\neg\Box p}(x)]$
$\wedge$	$\forall x(Q_{\Box q}(x) \rightarrow \forall y(R(x,y)\rightarrow Q_{q}(y))$	[from Sf6]
$\wedge$	$\forall x(Q_{\Box q}(x) \leftrightarrow \neg Q_{\neg\Box q}(x))$	$[Q_{\sim\Box q}(x) = Q_{\neg\Box q}(x)]$
$\wedge$	$\forall x(Q_{\neg\Box q}(x) \rightarrow \exists y(R(x,y)\wedge Q_{\neg q}(y))$	$[Q_{\sim\Box q}(x) = Q_{\neg\Box q}(x);\ Q_{\sim q}(y) = Q_{\neg q}(y)]$
$\wedge$	$\forall x(Q_{p}(x) \rightarrow \top)$	[from Sf5]
$\wedge$	$\forall x(Q_{p}(x) \leftrightarrow \neg Q_{\neg p}(x))$	$[Q_{\sim p}(x) = Q_{\neg p}(x)]$
$\wedge$	$\forall x(Q_{\neg p}(x) \rightarrow \neg Q_{p}(x))$	$[Q_{\sim p}(x) = Q_{\neg p}(x)]$
$\wedge$	$\forall x(Q_{q}(x) \rightarrow \top)$	[from Sf7]
$\wedge$	$\forall x(Q_{q}(x) \leftrightarrow \neg Q_{\neg q}(x))$	$[Q_{\sim q}(x) = Q_{\neg q}(x)]$
$\wedge$	$\forall x(Q_{\neg q}(x) \rightarrow \neg Q_{q}(x))$	$[Q_{\sim q}(x) = Q_{\neg q}(x)]$

Figure 2.3 **<u>Derivation of the simplification of the Definition for the axiomatic translation of Def(p)</u>**. This proof shows the simplification available for predicate p.

Def(p) =	$\forall x(Q_{p}(x) \rightarrow \pi(p, x))$	$\wedge$	$\forall x(Q_{\neg p}(x) \rightarrow \neg Q_{p}(x))$
$\wedge$	$\forall x(Q_{p}(x) \rightarrow \neg Q_{\sim p}(x))$		$[Q_{\sim p}(x)\rightarrow\neg Q_{p}(x) \equiv Q_{p}(x)\rightarrow\neg Q_{\sim p}(x)]$
$\wedge$	$\forall x(Q_{p}(x) \leftarrow \neg Q_{\sim p}(x))$	=	$\forall x(Q_{p}(x) \rightarrow \neg Q_{\neg p}(x))$
$\wedge$	$\forall x(Q_{\sim p}(x) \rightarrow \pi(\sim p, x))$	$\wedge$	$\forall x(Q_{p}(x) \leftarrow \neg Q_{\neg p}(x))$
=	$\forall x(Q_{p}(x) \rightarrow \top)$ [reduces to $\top$]	Def(p) =	$\forall x(Q_{p}(x) \rightarrow \neg Q_{\neg p}(x))$
$\wedge$	$\forall x(Q_{p}(x) \rightarrow \neg Q_{\neg p}(x))$	$\wedge$	$\forall x(Q_{p}(x) \leftarrow \neg Q_{\neg p}(x))$
$\wedge$	$\forall x(Q_{p}(x) \leftarrow \neg Q_{\neg p}(x))$		

Figure 2.4 **<u>Derivation of the simplification of the Definition for the axiomatic translation of Def(¬ψ)</u>**.The proof involves comparison of non-negated and negated standardized modal formulae, considering all three cases that might occur.

<u>Case 1.</u> For a modal formula ϕ:		=	$\forall x(Q_{\Box\phi}(x)\rightarrow\forall y(R(x, y) \rightarrow Q_{\phi}(y)))$
Def(ϕ) =	$\forall x(Q_{\phi}(x) \rightarrow \neg Q_{\neg\phi}(x))$	$\wedge$	$\forall x\ (Q_{\Box\phi}(x) \rightarrow \neg Q_{\neg\Box\phi}(x))$
$\wedge$	$\forall x(Q_{\phi}(x) \leftarrow \neg Q_{\neg\phi}(x))$	$\wedge$	$\forall x\ (Q_{\Box\phi}(x) \leftarrow \neg Q_{\neg\Box\phi}(x))$
Consider $\neg\phi$		$\wedge$	$\forall x(Q_{\neg\Box\phi}(x)\rightarrow\exists y(R(x,y) \wedge Q_{\neg\phi}(y)))$

$$\begin{aligned}
\mathrm{Def}(\neg\phi) &= \forall x(Q_{\neg\phi}(x) \rightarrow \pi(\neg\phi, x))\\
&\wedge \forall x(Q_{\neg\phi}(x) \rightarrow \neg Q_{\sim\neg\phi}(x))\\
&\wedge \forall x(Q_{\neg p}(x) \leftarrow \neg Q_{\sim\neg p}(x))\\
&\wedge \forall x(Q_{\sim\neg p}(x) \rightarrow \pi(\sim\neg p, x))\\
&= \forall x(Q_{\neg p}(x) \rightarrow \pi(\neg p, x))\\
&\wedge \forall x(Q_{\neg p}(x) \rightarrow \neg Q_{p}(x))\\
&\wedge \forall x(Q_{\neg p}(x) \leftarrow \neg Q_{p}(x))\\
&\wedge \forall x(Q_{p}(x) \rightarrow \pi(p, x))\\
&= \forall x(Q_{\neg\phi}(x) \rightarrow \neg Q_{\phi}(x))\\
&\wedge \forall x(Q_{\neg\phi}(x) \rightarrow \neg Q_{\phi}(x))\\
&\wedge \forall x(Q_{\neg\phi}(x) \leftarrow \neg Q_{\phi}(x))\\
&\wedge \forall x(Q_{\phi}(x) \rightarrow \top) \quad \text{[reduces to } \top\text{]}\\
&= \forall x(Q_{\neg\phi}(x) \rightarrow \neg Q_{\phi}(x))\\
&\wedge \forall x(Q_{\neg\phi}(x) \leftarrow \neg Q_{\phi}(x))\\
&= \forall x(Q_{\phi}(x) \rightarrow \neg Q_{\neg\phi}(x))\\
&\wedge \forall x(Q_{\phi}(x) \leftarrow \neg Q_{\neg\phi}(x))\\
&= \mathrm{Def}(\phi)
\end{aligned}$$

Case 2. Consider modal formula ϕ

$$\begin{aligned}
\mathrm{Def}(\Box\phi) &= \forall x\, (Q_{\Box\phi}(x) \rightarrow \pi(\Box\phi, x))\\
&\wedge \forall x\, (Q_{\Box\phi}(x) \rightarrow \neg Q_{\sim\Box\phi}(x))\\
&\wedge \forall x\, (Q_{\Box\phi}(x) \leftarrow \neg Q_{\sim\Box\phi}(x))\\
&\wedge \forall x\, (Q_{\sim\Box\phi}(x) \rightarrow \pi(\sim\Box\phi, x))\\
&= \forall x\, (Q_{\Box\phi}(x) \rightarrow \pi(\Box\phi, x))\\
&\wedge \forall x\, (Q_{\Box\phi}(x) \rightarrow \neg Q_{\neg\Box\phi}(x))\\
&\wedge \forall x\, (Q_{\Box\phi}(x) \leftarrow \neg Q_{\neg\Box\phi}(x))\\
&\wedge \forall x\, (Q_{\neg\Box\phi}(x) \rightarrow \pi(\neg\Box\phi, x))\\
&= \forall x(Q_{\Box\phi}(x) \rightarrow \forall y(R(x, y) \rightarrow Q_{\phi}(y)))\\
&\wedge \forall x\, (Q_{\Box\phi}(x) \rightarrow \neg Q_{\neg\Box\phi}(x))\\
&\wedge \forall x\, (Q_{\Box\phi}(x) \leftarrow \neg Q_{\neg\Box\phi}(x))\\
&\wedge \forall x(Q_{\neg\Box\phi}(x) \rightarrow \exists y(R(x,y) \wedge Q_{\neg\phi}(y)))
\end{aligned}$$

Consider

$$\begin{aligned}
\mathrm{Def}(\neg\Box\phi) &= \forall x\, (Q_{\neg\Box\phi}(x) \rightarrow \pi(\neg\Box\phi, x))\\
&\wedge \forall x\, (Q_{\neg\Box\phi}(x) \rightarrow \neg Q_{\sim\neg\Box\phi}(x))\\
&\wedge \forall x\, (Q_{\neg\Box\phi}(x) \leftarrow \neg Q_{\sim\neg\Box\phi}(x))\\
&\wedge \forall x\, (Q_{\sim\neg\Box\phi}(x) \rightarrow \pi(\sim\neg\Box\phi, x))\\
&= \forall x\, (Q_{\neg\Box\phi}(x) \rightarrow \pi(\neg\Box\phi, x))\\
&\wedge \forall x\, (Q_{\neg\Box\phi}(x) \rightarrow \neg Q_{\Box\phi}(x))\\
&\wedge \forall x\, (Q_{\neg\Box\phi}(x) \leftarrow \neg Q_{\Box\phi}(x))\\
&\wedge \forall x\, (Q_{\Box\phi}(x) \rightarrow \pi(\Box\phi, x))\\
&= \mathrm{Def}(\Box\phi)
\end{aligned}$$

Case 3. Consider modal formulae ϕ and ψ

$$\begin{aligned}
\mathrm{Def}(\phi\wedge\psi) &= \forall x(Q_{\phi\wedge\psi}(x) \rightarrow \pi(\phi\wedge\psi, x))\\
&\wedge \forall x(Q_{\phi\wedge\psi}(x) \rightarrow \neg Q_{\sim(\phi\wedge\psi)}(x))\\
&\wedge \forall x(Q_{\phi\wedge\psi}(x) \leftarrow \neg Q_{\sim(\phi\wedge\psi)}(x))\\
&\wedge \forall x(Q_{\sim(\phi\wedge\psi)}(x) \rightarrow \pi(\sim(\phi\wedge\psi), x))\\
&= \forall x(Q_{\phi\wedge\psi}(x) \rightarrow Q_{\phi}(x) \wedge Q_{\psi}(x))\\
&\wedge \forall x(Q_{\phi\wedge\psi}(x) \rightarrow \neg Q_{\neg(\phi\wedge\psi)}(x))\\
&\wedge \forall x(Q_{\phi\wedge\psi}(x) \leftarrow \neg Q_{\neg(\phi\wedge\psi)}(x))\\
&\wedge \forall x(Q_{\neg(\phi\wedge\psi)}(x) \rightarrow Q_{\neg\phi}(x) \vee Q_{\neg\psi}(x))
\end{aligned}$$

Consider

$$\begin{aligned}
\mathrm{Def}(\neg(\psi\wedge\phi)) &= \forall x(Q_{\neg(\phi\wedge\psi)}(x) \rightarrow \pi(\neg(\phi\wedge\psi), x))\\
&\wedge \forall x(Q_{\neg(\phi\wedge\psi)}(x) \rightarrow \neg Q_{\sim\neg(\phi\wedge\psi)}(x))\\
&\wedge \forall x(Q_{\neg(\phi\wedge\psi)}(x) \leftarrow \neg Q_{\sim\neg(\phi\wedge\psi)}(x))\\
&\wedge \forall x(Q_{\sim\neg(\phi\wedge\psi)}(x) \rightarrow \pi(\sim\neg(\phi\wedge\psi), x))\\
&= \forall x(Q_{\neg(\phi\wedge\psi)}(x) \rightarrow \pi(\neg(\phi\wedge\psi), x))\\
&\wedge \forall x(Q_{\neg(\phi\wedge\psi)}(x) \rightarrow \neg Q_{(\phi\wedge\psi)}(x))\\
&\wedge \forall x(Q_{\neg(\phi\wedge\psi)}(x) \leftarrow \neg Q_{(\phi\wedge\psi)}(x))\\
&\wedge \forall x(Q_{(\phi\wedge\psi)}(x) \rightarrow \pi((\phi\wedge\psi), x))
\end{aligned}$$

Figure 2.5. Example of an optimized axiomatic translation. The translation is given for the target formula $\varphi = \Box((\neg\Box(r,p)) \wedge \Box(r,q))$ that was previously seen in figure 2.2.

$$\begin{aligned}
\prod(\Box(\neg\Box p\wedge\Box q)) &= \exists x Q_{\Box(\neg\Box p\wedge\Box q)}(x)\\
&\wedge \forall x(Q_{\Box(\neg\Box p\wedge\Box q)}(x) \rightarrow \forall y(R(x,y)\rightarrow Q_{\neg\Box p\wedge\Box q}(y))) && \text{[from(1)]}\\
&\wedge \forall x(Q_{\Box(\neg\Box p\wedge\Box q)}(x) \leftrightarrow \neg Q_{\neg\Box(\neg\Box p\wedge\Box q)}(x))\\
&\wedge \forall x(Q_{\neg\Box(\neg\Box p\wedge\Box q)}(x) \rightarrow \exists y(R(x,y)\wedge Q_{\neg(\neg\Box p\wedge\Box q)}(y)))\\
&\wedge \forall x(Q_{\neg\Box p\wedge\Box q}(x) \rightarrow (Q_{\neg\Box p}(x)\wedge Q_{\Box q}(x))) && \text{[from(2)]}\\
&\wedge \forall x(Q_{\neg\Box p\wedge\Box q}(x) \leftrightarrow \neg Q_{\neg(\neg\Box p\wedge\Box q)}(x))\\
&\wedge \forall x(Q_{\neg(\neg\Box p\wedge\Box q)}(x) \rightarrow (Q_{\Box p}(x)\vee Q_{\neg\Box q}(x)))\\
&\wedge \forall x(Q_{\Box p}(x) \rightarrow \forall y(R(x,y)\rightarrow Q_{p}(y))) && \text{[from(4)]}\\
&\wedge \forall x(Q_{\Box p}(x) \leftrightarrow \neg Q_{\neg\Box p}(x))\\
&\wedge \forall x(Q_{\neg\Box p}(x) \rightarrow \exists y(R(x,y)\wedge Q_{\neg p}(y)))\\
&\wedge \forall x(Q_{\Box q}(x) \rightarrow \forall y(R(x,y)\rightarrow Q_{q}(y))) && \text{[from(6)]}\\
&\wedge \forall x(Q_{\Box q}(x) \leftrightarrow \neg Q_{\neg\Box q}(x))\\
&\wedge \forall x(Q_{\neg\Box q}(x) \rightarrow \exists y(R(x,y)\wedge Q_{\neg q}(y)))\\
&\wedge \forall x(Q_{p}(x) \leftrightarrow \neg Q_{\neg p}(x)) && \text{[from(5)]}\\
&\wedge \forall x(Q_{q}(x) \leftrightarrow \neg Q_{\neg q}(x)) && \text{[from(7)]}
\end{aligned}$$

2.2 Inclusion of modal axioms in the axiomatic translation of modal logic.

As already discussed, modal axioms are often formulated as additional constraints on the accessibility relations of Kripke frames in modal logic. The axioms are theorems of these new restricted systems.

2.2.1 The incorporation of the correspondence properties of common modal axioms.

The correspondence properties of several common axioms of modal logic are given in table 2.6. Different ways in which they can be derived has been sketched in section 1.4. Incorporation of these correspondence properties into the translation is simple. The correspondence properties from the table 2.6 are simply added by repeated conjugation ($\wedge$) to the list of translated formulae, as shown in figure 2.7

It is worth noting that many modal axioms do not have correspondence properties, but can be successfully subject to axiomatic translation, and to first-order resolution. A case in point is axiom M (McKinsey, $\Box\neg\Box\neg p\rightarrow\neg\Box\neg\Box p$). The schema encoding of this axiom is considered in the results section (section 7.2).

Table 2.6 Correspondence properties of common modal axioms. The formulae are modified from [1]. The correct accessibility relation (here, R) must be used in any translation.

Axiom	Correspondence Property (*Corr*)
T reflexive	$\forall x(R(x,x))$
D serial	$\forall x\exists y(R(x,y))$
B symmetry	$\forall xy(R(x,y) \rightarrow R(y,x))$
4 transitive	$\forall xyz((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$
5 Euclidean	$\forall xyz(R(x,y) \wedge R(x,z) \rightarrow R(y,z))$
alt_1 functional	$\forall xyz(R(x,y) \wedge R(x,z) \rightarrow (y\approx z))$
4^{κ} 4^2 4^3	$\forall xy(R^{\kappa+1}(x,y) \rightarrow R(x,y))$ $\forall xuyz((R(x,u) \wedge R(u,y) \wedge R(y,z)) \rightarrow R(x,z))$ $\forall xuvyz((R(x,u) \wedge R(u,v) \wedge R(v,y) \wedge R(y,z)) \rightarrow R(x,z))$
5^{κ} 5^2 5^3	$\forall xyz(R^{\kappa}(x,y) \wedge R(x,z) \rightarrow R(y,z))$ $\forall xuyz((R(x,u) \wedge R(u,y) \wedge R(x,z)) \rightarrow R(y,z))$ $\forall xuvyz((R(x,u) \wedge R(u,v) \wedge R(v,y) \wedge R(x,z)) \rightarrow R(y,z))$
$alt_1^{\kappa1,\kappa2}$ $alt_1^{1,1}$ $alt_1^{1,2}$ $alt_1^{2,1}$ $alt_1^{2,2}$	$\forall xyz(R^{\kappa1+1}(x,y) \wedge R^{\kappa2+1}(x,z) \rightarrow (y\approx z))$ $\forall xyz((\exists u(R(x,u) \wedge R(u,y)) \wedge \exists v(R(x,v) \wedge R(v,z))) \rightarrow (y\approx z))$ $\forall xyz((\exists u(R(x,u) \wedge R(u,y)) \wedge \exists vw(R(x,v) \wedge R(v,w) \wedge R(w,z))) \rightarrow (y\approx z))$ $\forall xyz((\exists uw(R(x,u) \wedge R(u,w) \wedge R(w,y)) \wedge \exists v(R(x,v) \wedge R(v,z))) \rightarrow (y\approx z))$ $\forall xyz((\exists uw(R(x,u)\wedge R(u,w)\wedge R(w,y))\wedge\exists vg(R(x,v)\wedge R(v,g)\wedge R(g,z)))\rightarrow(y\approx z))$

Figure 2.7 An example of the incorporation of correspondence properties in a translation. The example refers to the target formula $\Box((\neg\Box(r,p)) \wedge \Box(r,q))$ in KB_c

$\prod(\Box((\neg\Box(r,p))\wedge\Box(r,q)))$
$\wedge \quad \forall xy(R(x,y) \rightarrow R(y,x))$ [written in shorthand as *Corr*(B)]

2.2.2 **<u>The Incorporation of the schema encoding common modal axioms into axiomatic translation</u>** [1].

The schema encoding of common modal axioms is shown in table 2.8. The derivation of these translations is based upon normalization of the modal representation of the axiom by *limited* application of the relationships in formulae 2.3, in just the same way as has already been described in section 2.1. The derivations for the schema encodings are given in figure 2.9. As already seen, the manipulations terminate leaving all but the outermost modal operators in place (that is, all the inner modal operators in place). On the right hand side, there are still expressions in π, that need to be addressed. The arguments of these expressions in π are considered to be *ground terms* and so, a predicate may be substituted. For example, $\pi(q,x)$ is written $Q_q(x)$ and $\pi(\Box p,y)$ is written $Q_{\Box p}(y)$. It is worth noting that the schemas encoding the axioms are not deterministic. Hence, for example, with axiom D (figure 2.9), alternative forms of the Q-predicate symbols can be formulated. The schema chosen is the form in which the minimum number of new Q-predicates needs to be introduced. This has consequences for any algorithm for automatically generating these schemas: It will not be easy to ensure that these optimized forms are produced.

During translation, for each modal axiom (which is valid for a particular problem) a new formula is added ($\wedge$) to output. The origin of this formula is the encoding of an *edited* version the instantiation set, in which only sub-formulae immediately below a box ($\Box$) symbol are included. Each sub-formula in this edited-instantiation set is encoded by applying the transformations in table 2.8. The edited-instantiation set of φ is described by

$\Box\psi \in Sf(\varphi)$, and is denoted by $\chi^{\in}_{\varphi}$, where $\chi^{\in}_{\varphi} = \{ \psi \mid \Box\psi \in Sf(\varphi) \}$

and the superscript $^{\in}$ refers to the base state, with no modal axioms applied.

The schema for the particular modal axiom is *instantiated* for each member of the instantiation set. The process is best illustrated by an example; the translation of a target formula in KB seen in figure 2.10. Note that the notation is as follows: $B(\Box q)$ is an instance of the schema for axiom B instantiated for $\Box q$.

Table 2.8 also shows that, for the translation of some modal axioms, new symbols may need to be *Defined*, that is translated in $Def(\psi)$ (see formula 2.2), and the results again added ($\wedge$) to the output formula. The origin of these formulae is clear. For axiom D, the translation is $\neg Q_{\Box\phi}(x) \vee Q_{\neg\Box\neg\phi}(x)$. Here the predicate $Q_{\Box\phi}$ has been seen (and *Defined*) before, but the predicate $Q_{\neg\Box\neg\phi}$ is new. This arises from translation of the formula $\neg\Box\neg\phi$ in axiom D (recall, $\Box\phi \rightarrow \neg\Box\neg\phi$). It has already been shown (section 2.1) that the result of Definition of negated and non-negated formulae is identical, and hence it is the new formula $\Box\neg\phi$ that needs to be Defined. Making this clearer by considering a further example, if the sub-formula under consideration is $\Box\Box q$, then the new formula Defined is $Def(\Box\neg(\Box q))$ or $Def(\Box\neg\Box q)$. An example is shown in figure 2.11, for the translation of a target formula in KD. In a similar way, for axiom alt_1 ($\neg\Box p\rightarrow\Box\neg p$; $\neg Q_{\neg\Box\phi}(x)\vee Q_{\Box\neg\phi}(x)$), a new formula $Def(\Box\neg\phi)$ needs to be defined. For some modal axioms, no new formulae need to be defined (see table 2.8).

Table 2.8 **<u>Schema encoding of some modal axioms for axiomatic translation</u>**. (Adapted from figure 3 in [1]). Both the name of the axiom is given, and the corresponding geometrical property. In this table, the modality index r has been included. In the uni-modal case, this index is normally excluded.

There are other forms for the quantification of some of expressions, for example

$\forall xuyvz(\neg R(x,u) \vee \neg R(u,y) \vee \neg Q_{\neg\Box p}(y) \vee \neg R(x,v) \vee \neg R(v,z) \vee Q_{\Box\neg p}(z))$ for $Alt_1^{2,2}$.

Axiom	Schema encoding of $\Box(r,p)$	New symbols
T reflexive	$\forall x(\neg Q_{\Box r,p}(x) \vee Q_{r,p}(x))$	None
D serial	$\forall x(\neg Q_{\Box r,p}(x) \vee Q_{\neg\Box r,\neg p}(x))$	$\Box r,\neg p$
4 transitive	$\forall x(\neg Q_{\Box r,p}(x) \vee \forall y(\neg R(x,y) \vee Q_{\Box r,p}(y)))$	None
5 Euclidean	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box r,p}(y)) \vee Q_{\Box r,p}(x))$	None
B symmetry	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box r,p}(y)) \vee Q_{r,p}(x))$	None
alt_1 functional	$\forall x(\neg Q_{\neg\Box r,p}(x) \vee Q_{\Box r,\neg p}(x))$	$\Box r,\neg p$
4^{κ}	$\forall x(\neg Q_{\Box r,p}(x) \vee \forall y(\neg R^{\kappa}(x,y) \vee Q_{\Box r,p}(y)))$	None
4^2	$\forall x(\neg Q_{\Box r,p}(x) \vee \forall yz(\neg R(x,y) \vee \neg R(y,z) \vee Q_{\Box r,p}(z))))$	
4^3	$\forall x(\neg Q_{\Box r,p}(x) \vee \forall yzu(\neg R(x,y) \vee \neg R(y,z) \vee \neg R(z,u) \vee Q_{\Box r,p}(u)))$	
5^{κ}	$\forall x(Q_{\Box r,p}(x) \vee \forall y(\neg R^{\kappa}(x,y) \vee \neg Q_{\Box r,p}(y)))$	None
5^2	$\forall x(Q_{\Box r,p}(x) \vee \forall zy(\neg R(x,z) \vee \neg R(z,y) \vee \neg Q_{\Box r,p}(y)))$	
5^3	$\forall x(Q_{\Box r,p}(x) \vee \forall zuy(\neg R(x,z) \vee \neg R(z,u) \vee \neg R(u,y) \vee \neg Q_{\Box r,p}(y)))$	
$alt_1^{\kappa 1 \kappa 2}$	$\forall xyz(\neg R^{\kappa 1}(x,y)\vee\neg R^{\kappa 2}(x,z)\vee\neg Q_{\neg\Box r,p}(y)\vee Q_{\Box r,\neg p}(z))$	$\Box r,\neg p$
$alt_1^{1,1}$	$\forall x(\forall y[\neg R(x,y)\vee\neg Q_{\neg\Box r,p}(y)]\vee\forall z[\neg R(x,z)\vee Q_{\Box r,\neg p}(z)])$	
$alt_1^{1,2}$	$\forall x(\forall y[\neg R(x,y)\vee\neg Q_{\neg\Box r,p}(y)]\vee\forall uz[\neg R(x,u)\vee\neg R(u,z)\vee Q_{\Box r,\neg p}(z)])$	
$alt_1^{2,1}$	$\forall x(\forall uy[\neg R(x,u)\vee\neg R(u,y)\vee Q_{\neg\Box r,p}(y)]\vee\forall z[\neg R(x,z)\vee Q_{\Box r,\neg p}(z)])$	
$alt_1^{2,2}$	$\forall x(\forall uy[\neg R(x,u)\vee\neg R(u,y)\vee\neg Q_{\neg\Box r,p}(y)]\vee\forall vz[\neg R(x,v)\vee\neg R(v,z)\vee Q_{\Box r,\neg p}(z)])$	

Figure 2.9 **<u>Derivation of schema encoding of common axioms</u>**.

The formulae 2.4 are simply another (minimal) version of the definition already seen in formulae 2.3, but the modal formula p is universally quantified. π is the predicate ‘holds’ in [1]. p is any modal formula; x and y are distinct free variables; R is the accessibility relation.

$$\forall p\forall x(\pi(\neg p,x) \leftrightarrow \neg\pi(p, x)) \qquad [1]$$
$$\forall pq\forall x(\pi(p*q,x) \leftrightarrow (\pi(p, x)*\pi(q, x))) \qquad [2] \text{ where } * \in \{\rightarrow,\leftrightarrow,\vee,\wedge\}$$
$$\forall p\forall x(\pi(\Box p,x) \leftrightarrow \forall y(R(x,y) \rightarrow \pi(p, y))) \qquad [3] \qquad \text{[formulae 2.4]}$$

Axiom T: $\Box p \to p$

$\forall p \forall x(\pi(\Box p \to p,x)) \equiv \forall p \forall x(\pi(\Box p,x) \to \pi(p,x))$ [from 2.4, 2]

Re-writing (so $\pi(q,x)$ is $Q_q(x)$ and $\forall q$ is $\forall Q$, which can be taken as implicit) gives

$\forall x(Q_{\Box p}(x) \to Q_p(x)) \equiv$ **$\forall x(\neg Q_{\Box p}(x) \vee Q_p(x))$**

Axiom 4: $\Box p \to \Box\Box p$

$\forall p \forall x(\pi(\Box p \to \Box\Box p,x)$ $\equiv \forall p \forall x(\pi(\Box p,x) \to \pi(\Box\Box p,x))$ [from 2.4, 2]

$\equiv \forall p \forall x(\pi(\Box p,x) \to \forall y(R(x,y) \to \pi(\Box p,y)))$ [from 2.4, 3]

Re-writing gives $\forall x(Q_{\Box p}(x) \to \forall y(R(x,y) \to Q_{\Box p}(y))$

$\equiv$ **$\forall x(\neg Q_{\Box p}(x) \vee \forall y(\neg R(x,y) \vee Q_{\Box p}(y))$**

Axiom B: $\neg\Box\neg\Box p \to p$

$\forall p \forall x(\pi(\neg\Box\neg\Box p \to p,x)$ $\equiv \forall p \forall x(\pi(\neg\Box\neg\Box p,x) \to \pi(p,x))$ [from 2.4, 2]

$\equiv \forall p \forall x(\neg\pi(\Box\neg\Box p,x) \to \pi(p,x))$ [from 2.4, 1]

$\equiv \forall p \forall x(\neg\forall y(R(x,y) \to \pi(\neg\Box p,y)) \to \pi(p,x))$ [from 2.4, 3]

$\equiv \forall p \forall x(\neg\forall y(R(x,y) \to \neg\pi(\Box p,y)) \to \pi(p,x))$ [from 2.4, 1]

Re-writing gives $\forall x(\neg(\forall y(R(x,y) \to \neg Q_{\Box p}(y))) \to Q_p(x))$

$\equiv$ **$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y)) \vee Q_p(x))$**

Axiom D: $\Box p \to \neg\Box\neg p$

$\forall p \forall x(\pi(\Box p \to \neg\Box\neg p,x)$ $\equiv \forall p \forall x(\pi(\Box p,x) \to \pi(\neg\Box\neg p,x))$ [from 2.4, 2]

$\equiv \forall p \forall x(\pi(\Box p,x) \to \neg\pi(\Box\neg p,x))$ [from 2.4, 1]

Re-writing gives $\forall x(Q_{\Box p}(x) \to Q_{\neg\Box\neg p}(x))$

$\equiv$ **$\forall x(\neg Q_{\Box p}(x) \vee Q_{\neg\Box\neg p}(x))$**

or $\forall x(Q_{\Box p}(x) \to \neg Q_{\Box\neg p}(x))$

$\equiv$ **$\forall x(\neg Q_{\Box p}(x) \vee \neg Q_{\Box\neg p}(x))$**

Axiom alt$_1$: $\neg\Box p \to \Box\neg p$

$\forall p \forall x(\pi(\neg\Box p \to \Box\neg p,x)$ $\equiv \forall p \forall x(\pi(\neg\Box p,x) \to \pi(\Box\neg p,x))$ [from 2.4, 2]

Re-writing gives $\forall x(Q_{\neg\Box p}(x) \to Q_{\Box\neg p}(x))$

$\equiv$ **$\forall x(\neg Q_{\neg\Box p}(x) \vee Q_{\Box\neg p}(x))$**

Axiom 5: $\neg\Box\neg\Box p \to \Box p$

$\forall p \forall x(\pi(\neg\Box\neg\Box p \to \Box p,x)$ $\equiv \forall p \forall x(\pi(\neg\Box\neg\Box p,x) \to \pi(\Box p,x))$ [from 2.4, 2]

$\equiv \forall p \forall x(\neg\pi(\Box\neg\Box p,x) \to \pi(\Box p,x))$ [from 2.4, 1]

$\equiv \forall p \forall x(\neg\forall y(R(x,y) \to \pi(\neg\Box p,y)) \to \pi(\Box p,x))$ [from 2.4, 3]

$\equiv \forall p \forall x(\neg\forall y(R(x,y) \to \neg\pi(\Box p,y)) \to \pi(\Box p,x))$ [from 2.4, 1]

Re-writing gives $\forall x(\neg\forall y(R(x,y) \to \neg Q_{\Box p}(y)) \to Q_{\Box p}(x))$

$\equiv$ **$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y)) \vee Q_{\Box p}(x))$**

Axiom 4^K: $\Box p \to \Box^K\Box p$

$\forall p \forall x(\pi(\Box p \to \Box^K\Box p,x)$ $\equiv \forall p \forall x(\pi(\Box p,x) \to \pi(\Box^K\Box p,x))$ [from 2.4, 2]

κ=1 $\forall p \forall x(\pi(\Box p,x) \to \pi(\Box\Box p,x))$

$\equiv \forall p \forall x(\pi(\Box p,x) \to \forall y(R(x,y) \to \pi(\Box p,y)))$ [from 2.4, 3]

κ=2 $\forall p \forall x(\pi(\Box p,x) \to \pi(\Box\Box\Box p,x))$

$\equiv \forall p \forall x(\pi(\Box p,x) \to \forall y(R(x,y) \to \pi(\Box\Box p,y)))$ [from 2.4, 3]

$\equiv \forall p \forall x(\pi(\Box p,x) \to \forall y(R(x,y) \to \forall z(R(y,z) \to \pi(\Box p,z))))$ [from 2.4, 3]

κ=3 $\forall p \forall x(\pi(\Box p,x) \to \pi(\Box\Box\Box\Box p,x))$

$\equiv \forall p \forall x(\pi(\Box p,x) \to \forall y(R(x,y) \to \pi(\Box\Box\Box p,y)))$

$\equiv \forall p \forall x(\pi(\Box p,x) \to \forall y(R(x,y) \to \forall z(R(y,z) \to \pi(\Box\Box p,z))))$ [from 2.4, 3]

$\equiv \forall p \forall x(\pi(\Box p,x) \to \forall y(R(x,y) \to \forall z(R(y,z) \to \forall u(R(z,u) \to \pi(\Box p,u)))))$ [from 2.4, 3]

Re-writing gives

κ=1 $\forall x(Q_{\Box p}(x) \to \forall y(R(x,y) \to Q_{\Box p}(y)))$

$\equiv \forall x(\neg Q_{\Box p}(x) \vee \forall y(\neg R(x,y) \vee Q_{\Box p}(y)))$

κ=2 $\forall x(Q_{\Box p}(x) \to \forall y(R(x,y) \to \forall z(R(y,z) \to Q_{\Box p}(z))))$

$\equiv \forall x(\neg Q_{\Box p}(x) \vee \forall y(\neg R(x,y) \vee \forall z(\neg R(y,z) \vee Q_{\Box p}(z))))$

κ=3 $\forall x(Q_{\Box p}(x) \to \forall y(R(x,y) \to \forall z(R(y,z) \to \forall u(R(z,u) \to Q_{\Box p}(u)))))$

$\equiv \forall x(\neg Q_{\Box p}(x) \vee \forall y(\neg R(x,y) \vee \forall z(\neg R(y,z) \vee \forall u(\neg R(z,u) \vee Q_{\Box p}(u)))))$

κ **$\forall x(\neg Q_{\Box p}(x) \vee \forall y(\neg R^K(x,y) \vee Q_{\Box p}(y)))$**

Axiom 5^K: $\neg\Box^K\neg\Box p \to \Box p$

$\forall p \forall x(\pi(\neg\Box^K\neg\Box p \to \Box p,x) \equiv \forall p \forall x(\pi(\neg\Box^K\neg\Box p,x) \to \pi(\Box p,x))$ [from 2.4, 2]

κ=1 $\forall p \forall x(\pi(\neg\Box\neg\Box p,x) \to \pi(\Box p,x))$

$\equiv \forall p \forall x(\neg\pi(\Box\neg\Box p,x) \to \pi(\Box p,x))$ [from 2.4, 1]

$\equiv \forall p \forall x(\neg\forall y(R(x,y) \to \pi(\neg\Box p,y)) \to \pi(\Box p,x))$ [from 2.4, 3]

$\equiv \forall p \forall x(\neg\forall y(R(x,y) \to \neg\pi(\Box p,y)) \to \pi(\Box p,x))$ [from 2.4, 1]

κ=2 $\forall p \forall x(\pi(\neg\Box\Box\neg\Box p,x) \to \pi(\Box p,x))$

$\equiv \forall p \forall x(\neg\pi(\Box\Box\neg\Box p,x) \to \pi(\Box p,x))$ [from 2.4, 1]

$\equiv\ \forall p\forall x(\neg\forall y(R(x,y) \rightarrow \pi(\Box\neg\Box p,y)) \rightarrow \pi(\Box p,x))$ [from 2.4, 3]

$\equiv\ \forall p\forall x(\neg\forall y(R(x,y) \rightarrow \forall u(R(y,u) \rightarrow \pi(\neg\Box p,u))) \rightarrow \pi(\Box p,x))$ [from 2.4, 3]

$\equiv\ \forall p\forall x(\neg\forall y(R(x,y) \rightarrow \forall u(R(y,u) \rightarrow \neg\pi(\Box p,u))) \rightarrow \pi(\Box p,x))$ [from 2.4, 1]

$\kappa=3$ $\forall p\forall x(\pi(\neg\Box\Box\Box\neg\Box p,x) \rightarrow \pi(\Box p,x))$

$\equiv\ \forall p\forall x(\neg\pi(\Box\Box\Box\neg\Box p,x) \rightarrow \pi(\Box p,x))$ [from 2.4, 1]

$\equiv\ \forall p\forall x(\neg\forall y(R(x,y) \rightarrow \pi(\Box\Box\neg\Box p,y)) \rightarrow \pi(\Box p,x))$ [from 2.4, 3]

$\equiv\ \forall p\forall x(\neg\forall y(R(x,y) \rightarrow \forall u(R(y,u) \rightarrow \pi(\Box\neg\Box p,u))) \rightarrow \pi(\Box p,x))$ [from 2.4, 3]

$\equiv\ \forall p\forall x(\neg\forall y(R(x,y) \rightarrow \forall u(R(y,u) \rightarrow \forall v(R(u,v) \rightarrow \pi(\neg\Box p,v)))) \rightarrow \pi(\Box p,x))$ [from 2.4, 3]

$\equiv\ \forall p\forall x(\neg\forall y(R(x,y) \rightarrow \forall u(R(y,u) \rightarrow \forall v(R(u,v) \rightarrow \neg\pi(\Box p,v)))) \rightarrow \pi(\Box p,x))$ [from 2.4, 1]

Re-writing gives

$\kappa=1$ $\forall x(\neg\forall y(R(x,y) \rightarrow \neg Q_{\Box p}(y)) \rightarrow Q_{\Box p}(x))$

$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y)) \vee Q_{\Box p}(x))$

$\kappa=2$ $\forall x(\neg\forall y(R(x,y) \rightarrow \forall u(R(y,u) \rightarrow \neg Q_{\Box p}(u))) \rightarrow Q_{\Box p}(x))$

$\forall x(\forall y(\neg R(x,y) \vee \forall u(\neg R(y,u) \vee \neg Q_{\Box p}(u))) \vee Q_{\Box p}(x))$

$\kappa=3$ $\forall x(\neg\forall y(R(x,y) \rightarrow \forall u(R(y,u) \rightarrow \forall v(R(u,v) \rightarrow \neg Q_{\Box p}(v)))) \rightarrow Q_{\Box p}(x))$

$\forall x(\forall y(\neg R(x,y) \vee \forall u(\neg R(y,u) \vee \forall v(\neg R(u,v) \vee \neg Q_{\Box p}(v)))) \vee Q_{\Box p}(x))$

κ $\boldsymbol{\forall x(\forall y(\neg R^{K}(x,y) \vee \neg Q_{\Box p}(y)) \vee Q_{\Box p}(x))}$

<u>Axiom alt$_1^{K1,K2}$</u>: $\neg\Box^{K1}\Box p \rightarrow \Box^{K2}\Box\neg p$

$\forall p\forall x(\pi(\neg\Box^{K1}\Box p \rightarrow \Box^{K2}\Box\neg p,x) \equiv \forall p\forall x(\pi(\neg\Box^{K1}\Box p,x) \rightarrow \pi(\Box^{K2}\Box\neg p,x))$ [from 2.4, 2]

$\kappa1=1, \kappa2=1$ $\forall p\forall x(\pi(\neg\Box\Box p,x) \rightarrow \pi(\Box\Box\neg p,x))$

$\equiv\ \forall p\forall x(\neg\pi(\Box\Box p,x) \rightarrow \pi(\Box\Box\neg p,x))$ [from 1]

$\equiv\ \forall p\forall x(\neg\forall y(R(x,y) \rightarrow \pi(\Box p,y)) \rightarrow \forall z(R(x,z) \rightarrow \pi(\Box\neg p,z)))$ [from 2.4, 3]

$\kappa1=1, \kappa2=2$ $\forall p\forall x(\pi(\neg\Box\Box p,x) \rightarrow \pi(\Box\Box\Box\neg p,x))$

$\equiv\ \forall p\forall x(\neg\forall y(R(x,y) \rightarrow \pi(\Box p,y)) \rightarrow \forall u(R(x,u) \rightarrow \pi(\Box\Box\neg p,u)))$ [from 2.4, 1 & 3]

$\equiv\ \forall p\forall x(\neg\forall y(R(x,y) \rightarrow \pi(\Box p,y)) \rightarrow \forall u(R(x,u) \rightarrow \forall z(R(u,z) \rightarrow \pi(\Box\neg p,z))))$ [from 2.4, 3]

$\kappa1=2, \kappa2=1$ $\forall p\forall x(\pi(\neg\Box\Box\Box p,x) \rightarrow \pi(\Box\Box\neg p,x))$

$\equiv\ \forall p\forall x(\neg\forall u(R(x,u) \rightarrow \pi(\Box\Box p,u)) \rightarrow \forall z(R(x,z) \rightarrow \pi(\Box\neg p,z)))$ [from 2.4, 1 & 3]

$\equiv\ \forall p\forall x(\neg\forall u(R(x,u) \rightarrow \forall y(R(u,y) \rightarrow \pi(\Box p,y))) \rightarrow \forall z(R(x,z) \rightarrow \pi(\Box\neg p,z)))$ [from 2.4, 3]

$\kappa1=2, \kappa2=2$ $\forall p\forall x(\pi(\neg\Box\Box\Box p,x) \rightarrow \pi(\Box\Box\Box\neg p,x))$

$\equiv\ \forall p\forall x(\neg\forall u(R(x,u) \rightarrow \pi(\Box\Box p,u)) \rightarrow \forall v(R(x,v) \rightarrow \pi(\Box\Box\neg p,v)))$ [from 2.4, 1 & 3]

$\equiv\ \forall p\forall x(\neg\forall u(R(x,u) \rightarrow \forall y(R(u,y) \rightarrow \pi(\Box p,y))) \rightarrow \forall v(R(x,v) \rightarrow \forall z(R(v,z) \rightarrow \pi(\Box\neg p,z))))$ [from 3]

Re-writing gives

$\kappa1=1, \kappa2=1$ $\forall x(\neg\forall y(R(x,y) \rightarrow Q_{\Box p}(y)) \rightarrow \forall z(R(x,z) \rightarrow Q_{\Box\neg p}(z)))$

$\equiv\ \forall x(\forall y(\neg R(x,y) \vee Q_{\Box p}(y)) \vee \forall z(\neg R(x,z) \vee Q_{\Box\neg p}(z)))$

$\kappa1=1, \kappa2=2$ $\forall x(\neg\forall y(R(x,y) \rightarrow Q_{\Box p}(y)) \rightarrow \forall u(R(x,u) \rightarrow \forall z(R(u,z) \rightarrow Q_{\Box\neg p}(z))))$

$\equiv\ \forall x(\forall y(\neg R(x,y) \vee Q_{\Box p}(y)) \vee \forall u(\neg R(x,u) \vee \forall z(\neg R(u,z) \vee Q_{\Box\neg p}(z))))$

$\kappa1=2, \kappa2=1$ $\forall x(\neg\forall u(R(x,u) \rightarrow \forall y(R(u,y) \rightarrow Q\Box p(y))) \rightarrow \forall z(R(x,z) \rightarrow Q\Box\neg p(z)))$

$\equiv\ \forall x(\forall u(\neg R(x,u) \vee \forall y(\neg R(u,y) \vee Q\Box p(y))) \vee \forall z(\neg R(x,z) \vee Q\Box\neg p(z)))$

$\kappa1=2, \kappa2=2$ $\forall x(\neg\forall u(R(x,u) \rightarrow \forall y(R(u,y) \rightarrow Q_{\Box p}(y))) \rightarrow \forall v(R(x,v) \rightarrow \forall z(R(v,z) \rightarrow Q_{\Box\neg p}(z))))$

$\equiv\ \forall x(\forall u(\neg R(x,u) \vee \forall y(\neg R(u,y) \vee Q_{\Box p}(y))) \vee \forall v(\neg R(x,v) \vee \forall z(\neg R(v,z) \vee Q_{\Box\neg p}(z))))$

$\kappa1,\kappa2$ $\boldsymbol{\forall x(\forall y(\neg R^{K1}(x,y) \vee Q_{\Box p}(y)) \vee \forall z(\neg R^{K2}(x,z) \vee Q_{\Box\neg p}(z)))}$

Figure 2.10 <u>The axiomatic translation of a target formula in axiom B</u>. The translation for the target formula $\varphi = \Box((\neg\Box(r,p)) \wedge \Box(r,q))$ in KB is shown, with the edited instantiation set is { $\boldsymbol{\Box((\neg\Box p)\wedge\Box q)}$, $\boldsymbol{\Box p}$, $\boldsymbol{\Box q}$ }. The schema encoding of B is $\forall x(\forall y(\neg R(x,y)\vee\neg Q_{\Box r,p}(y)) \vee Q_{r,p}(x))$. This is a uni-modal case and so modality index r can be safely dropped; R is the predicate symbol associated with the modality index.

The translation of φ in KB :

	$\prod(\Box((\neg\Box p)\wedge\Box q))$	
$\wedge$	$\forall x(\forall y(\neg R(x,y)\vee\neg Q_{\Box(\neg\Box p\wedge\Box q)}(y)) \vee Q_{\neg\Box p\wedge\Box q}(x))$	B($\Box(\neg\Box p\wedge\Box q)$)
$\wedge$	$\forall x(\forall y(\neg R(x,y)\vee\neg Q_{\Box p}(y)) \vee Q_{p}(x))$	B($\Box p$)
$\wedge$	$\forall x(\forall y(\neg R(x,y)\vee\neg Q_{\Box q}(y)) \vee Q_{q}(x))$	B($\Box q$)

Figure 2.11 The axiomatic translation of a target formula in axiom D.

The translation for the target formula $\varphi = \Box((\neg\Box(r,p)) \wedge \Box(r,q))$ in KD is shown. Note two new symbols need to be Defined.

$$
\begin{array}{lll}
 & \prod(\Box((\neg\Box p)\wedge\Box q)) & \\
\wedge & \forall x(\neg Q_{\Box(\neg\Box p\wedge\Box q)}(x) \vee Q_{\neg\Box\neg(\neg\Box p\wedge\Box q)}(x)) & [D(\Box(\neg\Box p\wedge\Box q))] \\
\wedge & \forall x(\neg Q_{\Box p}(x) \vee Q_{\neg\Box\neg p}(x)) & [D(\Box p)] \\
\wedge & \forall x(\neg Q_{\Box q}(x) \vee Q_{\neg\Box\neg q}(x)) & [D(\Box q)] \\
\wedge & \forall x(Q_{\Box\neg(\neg\Box p\wedge\Box q)}(x) \rightarrow \forall y(R(x,y) \rightarrow Q_{\neg(\neg\Box p\wedge\Box q)}(y))) & [Def(\Box\neg(\neg\Box p\wedge\Box q))] \\
\wedge & \forall x(Q_{\Box\neg(\neg\Box p\wedge\Box q)}(x) \leftrightarrow \neg Q_{\neg\Box\neg(\neg\Box p\wedge\Box q)}(x)) & \\
\wedge & \forall x(Q_{\neg\Box\neg(\neg\Box p\wedge\Box q)}(x) \rightarrow \exists y(R(x,y) \wedge Q_{\neg\Box p\wedge\Box q}(y))) & \\
\wedge & \forall x(Q_{\Box\neg p}(x) \rightarrow \forall y(R(x,y) \rightarrow Q_{\neg p}(y))) & [Def(\Box\neg p)] \\
\wedge & \forall x(Q_{\Box\neg p}(x) \leftrightarrow \neg Q_{\neg\Box\neg p}(x)) & \\
\wedge & \forall x(Q_{\neg\Box\neg p}(x) \rightarrow \exists y(R(x,y) \wedge Q_{p}(y))) & \\
\wedge & \forall x(Q_{\Box\neg q}(x) \rightarrow \forall y(R(x,y) \rightarrow Q_{\neg q}(y))) & [Def(\Box\neg q)] \\
\wedge & \forall x(Q_{\Box\neg q}(x) \leftrightarrow \neg Q_{\neg\Box\neg q}(x)) & \\
\wedge & \forall x(Q_{\neg\Box\neg q}(x) \rightarrow \exists y(R(x,y) \wedge Q_{q}(y))) &
\end{array}
$$

2.2.3 Composition in the incorporation of several modal axioms in the axiomatic translation. (See section 5 in [1]).

It is commonly necessary for several modal axioms to be valid in a restricted set of Kripke frames in order to model a useful property. How are these combinations of axioms handled in axiomatic translation? In many cases, formulae arising from the translation of a set of sub-formulae are just added ($\wedge$) together in the final output formula with no interaction between the modal axioms in the translation process. This is the case for combinations of axioms: KT4, KTB, KDB, KD4 (for proofs, see [1]). Importantly, note that since there is no interaction between the modal axioms during the translation, the *order* in which these modal axioms are applied to a set of sub-formulae is *unimportant*. An example of the translation of a target formula in KTB is shown in figure 2.12, and more interestingly of a target formula in KDB in figure 2.13.Note in that while the new formulae arising from axiom B need to be Defined (as described above), they are *not* added to the instantiation set considered when axiom B is applied.

Unfortunately, this simple procedure does not represent the general case. The combinations of axioms K4B and KT4B require *composition* of axiom 4 (see below). First it is worth considering the cases K5 and $K5^{\kappa}$ that have complications that are related to this problem of composition. For axiom 5, the base-case of the instantiation set ($\chi^{\in}_{\varphi}$) of sub-formula immediately below a box ($\Box$) symbol is *not sufficient* to ensure a complete translation. An additional term of form $\Box\neg\Box\psi$ for each sub-formula $\Box\psi$ must (i) be included in the instantiation set, and (ii) Defined, so:

$$\Box\chi^{5}_{\varphi} = \Box\chi^{\in}_{\varphi} \cup \Box\neg\Box\chi^{\in}_{\varphi}$$

An example of the translation of a target formula in K5 is given in figure 2.14. (Note, that it is sometimes convenient to refer to the default translation of axiom 5 as $K5_o$ in order to highlight the complication that exists here; if this terminology is being used then K5 represents the translation without these compositional formulae. This translation K5 may be used in experiments – but is not complete under resolution and so may not give the correct answer for the translation of any particular target modal formula).

In many cases, the satisfiability of a target formula is not affected if these additional sub-formula are not included in the instantiation set (as in the example in figure 2.14, which is satisfiable for both K5 and $K5_o$), but a case from [1] that discriminates has be included in the test suite: $\neg\Box\Box\Box p \wedge \Box\Box p$ (in K5 satisfiable; but in $K5_o$ unsatisfiable). For this case :

$\Box\chi^5_\varphi$ is $\{\Box\Box\Box p, \Box\Box p, \Box p\}$, and

$Sf(\varphi) = \{\neg\Box\Box\Box p \wedge \Box\Box p, \mathit{\Box\Box\Box p, \Box\Box p, \Box p}, p\}$

and the translation is :

$\prod(\neg\Box\Box\Box p \wedge \Box\Box p)$

$\wedge\ 5(\Box\Box\Box p) \wedge 5(\Box\Box p) \wedge 5(\Box p)$

$\wedge\ Def(\Box\neg\Box\Box\Box p) \wedge\ Def(\Box\neg\Box\Box p) \wedge\ Def(\Box\neg\Box p)$

$\wedge\ 5(\Box\neg\Box\Box\Box p) \wedge\ 5(\Box\neg\Box\Box p) \wedge\ 5(\Box\neg\Box p)$

During this study other *counterexamples* for translations without the additional compositional formulae were identified (see figure 7.10).

Translation of axiom 5 is further illustrated by the example in figure 2.15 where the origin of various subformulae is explicitly traced for a translation in $K5_oT$. Take note of the update to the instantiation set that occurs for composition of axiom 5_o. Similar considerations apply to axiom 5^κ (see table 2.8). A translation in axiom 5^κ is shown in figure 2.16. A new feature of the axiomatic translation is seen here. The additional formulae marked * in figure 2.16 arise by considering all the sub-formulae in the new compositional formulae. In the case of axiom 5, no additional clauses arise from considering all the sub-formulae of the newly defined formulae (only $\Box\neg\Box\psi$ arises from $\Box\psi$). In the case of axiom 5^2 and axiom 5^3, a *series* of new formulae arise. For 5^2 both $\Box\Box\neg\Box\psi$ and $\Box\neg\Box\psi$ and arise from $\Box\psi$, and for 5^3 all of $\Box\Box\Box\neg\Box\psi, \Box\Box\neg\Box\psi$ and $\Box\neg\Box\psi$ arise from $\Box\psi$. It is possible to view the compositional terms for axioms 5 and 5^κ to be an *internal* composition, isolated within the axiom schema itself.

Composition can also take place for sequences of modal axioms: New compositional formulae in the translation arise from the interactions between axioms.

Hence, for the axiom combination K4B, the base-case of the instantiation set ($\chi^{\in}_{\varphi}$) is not sufficient. An additional formula of form $\Box\Box\psi$ for each sub-formula $\Box\psi$ must be included. The composition of a series of modal axioms can be described in formulae of χ^{α}_{φ} as shown in formula 2.5 [taken from 1].

$$\chi^{\in}_{\varphi} = \{ \psi \mid \Box\psi \in Sf(\varphi) \}$$

$$\chi^{\alpha.A}_{\varphi} = \chi^{\alpha}_{\varphi} \cup \{ \phi\{p/\psi\} \mid \Box\phi \in \Box\chi^{\in}_{A}, \Box\psi \in Sf(\Box\chi^{\alpha}_{\varphi}) \}$$

(formulae 2.5, see [1])

In formula 2.5, φ is a modal formula; $\Box\psi$ is the set of box formulae in the sub formulae of φ, and ψ is the same set without the leading box; $\Box\chi^{\in}_{\varphi}$ is the base instantiation set with $\Box$ at the top of the formula, with no axioms applied, where $\in$ is the empty sequence of axioms (that is, just axiom K); $\Box\chi^{\alpha}_{\varphi}$ is the instantiation set, plus formulae induced by the axioms α, where α is an *ordered* sequence of axioms; $\Box\chi^{\alpha.A}_{\varphi}$ is the instantiation set, with the sequence of modal axioms α and then axiom A, in that order; and the substitution refers to the free variable p in formulae in table 2.8.

There is a difference from the internal composition described for axioms $5_o/5_o^{\kappa}$. Here multiple axioms are being applied in sequence, and the translation of an axiom may influence the instantiation set only for the *next* axiom. That is, for axioms 5 and 5^K, the update of the instantiation set occurs *before* translation of the axiom (and effects both the current axiom $5/5^{\kappa}$ *and* the subsequent axioms), and in other cases (axioms T, B, D, 4, 4^{κ}, alt_1, $alt_1^{\kappa1,\kappa2}$) the update of the instantiation set only effects the application of *subsequen*t axioms (*not* the current axiom). For the case of axiom combination K4B, the instantiation sets for the translation are $\chi_4=\chi^{\in}_{\varphi}$ and $\chi_B=\chi^{4}_{\varphi}$ [table 2.17]. An example that has been included in the test suite is $\varphi=\neg(\Box\neg\Box p\vee\Box p)$ in K4B [mentioned in 1 as discriminating between complete and non-complete translations, of the format $K4_oB$ and K4B respectively]. Here

$Sf(\varphi) = \{ \neg\Box\neg\Box p\wedge\neg\Box p, \Box\neg\Box p, \Box p, p \}$, and $\Box\chi = \{ \Box\neg\Box p, \Box p \}$

and the translation is

$\Pi(\Box\neg\Box p\vee\Box p)$

$\wedge\ 4(\Box\neg\Box p) \wedge 4(\Box p) \wedge Def(\Box\Box\neg\Box p) \wedge Def(\Box\Box p)$ **

$\wedge\ B(\Box\neg\Box p) \wedge B(\Box p) \wedge B(\Box\Box\neg\Box p) \wedge B(\Box\Box p)$

Considering the sequence of transformations that take place, at point **, the instantiation set is updated to include the new formulae as follows: $\Box\chi = \{\Box\neg\Box p, \Box p, \Box\Box\neg\Box p, \Box\Box p\}$. How is this derived? For this example, in formulae 2.5; $\chi^{\in}_{\varphi} = \{ \neg\Box p,\ p \}$; $\Box\chi^{\in}_{\varphi} = \{ \Box\neg\Box p, \Box p \}$; $\psi = \{ \neg\Box p, p \}$; and $\Box\psi = \{ \Box\neg\Box p, \Box p \}$. $\Box\psi$, the instantiation set when axiom 4 is translated (that is, $\chi_4 = \chi^{\in}_{\varphi}$). However, the instantiation set must be updated at the point ** *in light of* axiom 4. Then

$$\chi^{4}_{\varphi} = \chi^{\in.4}_{\varphi} = \chi^{\in}_{\varphi} \cup \{ \phi\,\{p/\psi\} \mid \Box\phi \in \Box\chi^{\in}_{4}, \Box\psi \in Sf(\Box\chi^{\in}_{\varphi}) \}.$$

It is this instantiation set that is used during the translation of axiom B (that is, $\chi_B = \chi^{4}_{\varphi}$). To construct this instantiation set, it is necessary to recall that axiom 4 is: $\neg\Box p \vee \Box\Box p$. Substituting ψ in $\Box\chi^{\varepsilon}_{4}$, i.e. $\neg\Box\psi\vee\Box\Box\psi$ gives:

$\neg\Box(\neg\Box p) \vee \Box\Box(\neg\Box p)$ giving rise to $\{\Box(\neg\Box p), \Box\Box(\neg\Box p)\}$

$\neg\Box(p) \vee \Box\Box(p)$ giving rise to $\{\Box(p), \Box\Box(p)\}$

so $\Box\phi \in \{ \Box\neg\Box p, \Box\Box\neg\Box p, \Box p, \Box\Box p \}$ and $\phi \in \{ \neg\Box p, \Box\neg\Box p, p, \Box p \}$

and $\chi^{4.}_{\varphi} = \{ \neg\Box p, p \} \cup \{ \neg\Box p, \Box\neg\Box p, p, \Box p \} = \{ \neg\Box p, \Box\neg\Box p, p, \Box p \}$

Finally, $\Box\chi^{4}_{\varphi} = \{ \Box\neg\Box p, \Box\Box\neg\Box p, \Box p, \Box\Box p \}$, in which the new formulae are $\{ \Box(\Box\neg\Box p), \Box(\Box p) \}$ arising from $\Box\psi = \{\Box\neg\Box p, \Box p \}$ respectively.

In the general case new formulae arising are of form $\Box\Box\psi$ for each sub-formula $\Box\psi$.

It can be seen that these new formulae need to be both *Defined*, and also appended to the instantiation set for the translation of axiom B (that is, $\chi_B = \chi^{4}_{\varphi}$).

Note that it is sometimes convenient to refer to the default translation of axiom sequence K4B as $K4_oB$ to denote compostion It should also be apparent that there is considerable scope for expressions to be re-Defined during the manipulations described above, and a simple extension to the translation procedure, is to define a *set* of formulae, so that there is no duplication in effort (that is, additional formulae added needlessly to the first-order formula). If such duplication were allowed, it would be likely to slow down or extend the proof by resolution. Other counter-examples for the combination of axioms $K4_oB$ were identified in this study (see figure 7.10). Similar considerations apply to the combination $KT4_oB$ (see figure 7.10 for counter-examples).

Other combinations of axioms, and the default instantiation sets required to ensure completeness have been described in [1]. They are listed in table 2.17. In [1], proofs are offered for the instantiation sets listed in table 2.17, and in some cases the positive shortcuts (section 2.2.4) that are usually considered optional must be included in the translation in order to ensure completeness.

It appears that the combination of axioms has a more general result (can be inferred from [1], but not proven). Simply that it is possible to define a compositional mechanism specifically tuned to each of the modal axioms, in which new sub-formulae are added to the instantiation set as each modal axiom is applied. Clearly, the *order* in which axioms are applied is important. The instantiation of an axiom schema encoding must process the compositional sub-formulae arising from axioms applied *before* it, and will in turn contribute new sub-formulae that need to be processed by the axioms that are applied *after* it. Table 2.18 lists these new sub-formulae associated with each modal axiom in composition [some are given in [1], others are derived here]. Axioms 4_o and 5_o appear as already described. It should be clear that for instantiation of the axiom sequence $K\alpha_1 5_o \alpha_2$, the instantiation set is updated with new compositional formulae *before* instantiation of axiom 5, and the subsequent axioms (α_2) also need to take into account the extra formulae (in the form of $\Box\neg\Box p$ added formulae to the instantiation set). A sketch of some of the derivations of table 2.18 is given in figure 2.19. Only a limited number are given, since it soon becomes apparent that deriving the general case is very mechanical, requiring only inspection of the modal form of the axiom and searching for formulae other than those of the form $\Box p$ and p. Hence axiom D is : $\Box p \rightarrow \neg\Box\neg p$ and eliminating $\Box p$, leaves $\Box\neg p$ for the compositional formulae (recall that the state of negation of these subformulae is irrelevant).

It is worth noting that in principle, all axiom combinations could use composition, with additional formulae in the instantiation set arising from previously applied axioms. The result of resolution would be identical for combinations like KT4 and KTB (trivially equal because there are no compositional formulae for axiom T), and KD_oB = KDB and KD_o4 = KD4. The inclusion of more subformulae (that is, *unnecessary* subformulae) in the translation passed to the SPASS resolution prover extends the problem to be considered, and may therefore lengthen the proof procedure.

Figure 2.12 **The axiomatic translation of a target formula in axioms T and B**.

The translation for the target formula $\varphi = \Box((\neg\Box(r,p)) \wedge \Box(r,q))$ in KTB is shown.

	$\prod(\Box(\neg\Box p\wedge\Box q))$	
$\wedge$	$\forall x(\neg Q_{\Box(\neg\Box p\wedge\Box q)}(x) \vee Q_{\neg\Box p\wedge\Box q}(x))$	$T(\Box(\neg\Box p\wedge\Box q))$
$\wedge$	$\forall x(\neg Q_{\Box p}(x) \vee Q_{p}(x))$	$T(\Box p)$
$\wedge$	$\forall x(\neg Q_{\Box q}(x) \vee Q_{q}(x))$	$T(\Box q)$
$\wedge$	$\forall x(\forall y(\neg R(x,y)\vee\neg Q_{\Box(\neg\Box p\wedge\Box q)}(y)) \vee Q_{\neg\Box p\wedge\Box q}(x))$	$B(\Box(\neg\Box p\wedge\Box q))$
$\wedge$	$\forall x(\forall y(\neg R(x,y)\vee\neg Q_{\Box p}(y)) \vee Q_{p}(x))$	$B(\Box p)$
$\wedge$	$\forall x(\forall y(\neg R(x,y)\vee\neg Q_{\Box q}(y)) \vee Q_{p}(x))$	$B(\Box q)$

Figure 2.13 **The axiomatic translation of a target formula in axioms D and B**.

The translation for the target formula $\varphi = \Box((\neg\Box(r,p)) \wedge \Box(r,q))$ in KDB is shown.

* Note that while the new formulae arising from axiom B need to be Defined, they are *not* added to the instantiation set considered when axiom B is applied.

	$\prod(\Box(\neg\Box p\wedge\Box q))$	
$\wedge$	$\forall x(\neg Q_{\Box(\neg\Box p\wedge\Box q)}(x) \vee Q_{\neg\Box\neg(\neg\Box p\wedge\Box q)}(x))$	$D(\Box(\neg\Box p\wedge\Box q))$
$\wedge$	$\forall x(\neg Q_{\Box p}(x) \vee Q_{\neg\Box\neg p}(x))$	$D(\Box p)$
$\wedge$	$\forall x(\neg Q_{\Box q}(x) \vee Q_{\neg\Box\neg q}(x))$	$D(\Box q)$
$\wedge$	$\mathrm{Def}(\Box\neg(\neg\Box p\wedge\Box q)) \wedge \mathrm{Def}(\Box\neg p) \wedge \mathrm{Def}(\Box\neg q)$	*
$\wedge$	$\forall x(\forall y(\neg R(x,y)\vee\neg Q_{\Box(\neg\Box p\wedge\Box q)}(y)) \vee Q_{\neg\Box p\wedge\Box q}(x))$	$B(\Box(\neg\Box p\wedge\Box q))$
$\wedge$	$\forall x(\forall y(\neg R(x,y)\vee\neg Q_{\Box p}(y)) \vee Q_{p}(x))$	$B(\Box p)$
$\wedge$	$\forall x(\forall y(\neg R(x,y)\vee\neg Q_{\Box q}(y)) \vee Q_{p}(x))$	$B(\Box q)$

Figure 2.14 **The axiomatic translation of a target formula in axiom 5**.

The translation for the target formula $\varphi = \Box((\neg\Box(r,p)) \wedge \Box(r,q))$ in K5 is shown.

	$\prod(\Box(\neg\Box p\wedge\Box q))$	
$\wedge$	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box(\neg\Box p\wedge\Box q)}(y)) \vee Q_{\Box(\neg\Box p\wedge\Box q)}(x)) \wedge \mathrm{Def}(\Box\neg\Box(\neg\Box p\wedge\Box q))$	[$5(\Box(\neg\Box p\wedge\Box q))$]
$\wedge$	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box\neg\Box(\neg\Box p\wedge\Box q)}(y)) \vee Q_{\Box\neg\Box(\neg\Box p\wedge\Box q)}(x))$	[$5(\Box\neg\Box(\neg\Box p\wedge\Box q))$ from $\Box(\neg\Box p\wedge\Box q)$]
$\wedge$	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y)) \vee Q_{\Box p}(x)) \wedge \mathrm{Def}(\Box\neg\Box p)$	[$5(\Box p)$]
$\wedge$	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box\neg\Box p}(y)) \vee Q_{\Box\neg\Box p}(x))$	[$5(\Box\neg\Box p)$ from $\Box p$]
$\wedge$	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box q}(y)) \vee Q_{\Box q}(x)) \wedge \mathrm{Def}(\Box\neg\Box q)$	[$5(\Box q)$]
$\wedge$	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box\neg\Box q}(y)) \vee Q_{\Box\neg\Box q}(x))$	[$5(\Box\neg\Box q)$ from $\Box q$]

Figure 2.15 <u>The axiomatic translation of a target formula in axiom combination $K5_oT$</u>. Formulae contributing to translation of axiom 5, those making this translation complete for 5o, and the contribution of new compositional formulae added to the instantiation set by the incorporation of axiom T are highlighted.

Input … $\neg(\Box(r,p)\rightarrow p)$ in $K5_oT$

Standardized … $\Box(r,p)\wedge\neg p$ or $\wedge(\Box(r,p),\neg p)$

Instantiation set … { ***$\Box(r,p)$***, p, $\Box(r,p)\wedge\neg p$ }

$Def(\Box(r,p))$	$\forall x(Q_{\Box rp}(x) \leftrightarrow \neg Q_{\neg\Box rp}(x))$	K
	$\wedge\ \forall x(Q_{\Box rp}(x) \rightarrow \forall y(R_r(x,y) \rightarrow Q_p(y)))$	
	$\wedge\ \forall x(Q_{\neg\Box rp}(x) \rightarrow \exists y(R_r(x,y) \wedge Q_{\neg p}(y)))$	
$Def(\wedge(\Box(r,p),\neg(p)))$	$\wedge\ \forall x(Q_{\wedge\Box rp\neg p}(x) \leftrightarrow \neg Q_{\neg\wedge\Box rp\neg p}(x))$	
	$\wedge\ \forall x(Q_{\wedge\Box rp\neg p}(x) \rightarrow (Q_{\neg p}(x) \wedge Q_{\Box rp}(x)))$	
	$\wedge\ \forall x(Q_{\neg\wedge\Box rp\neg p}(x) \rightarrow (Q_p(x) \vee Q_{\neg\Box rp}(x)))$	
$Def(p)$	$\wedge\ \forall x(Q_p(x) \leftrightarrow \neg Q_{\neg p}(x))$	
□ Instantiation set … { $\Box(r,p)$ }		5 / 5o
$5(\Box(r,p))$	$\wedge\ \forall x(Q_{\Box rp}(x) \vee \forall y(\neg R_r(x,y) \vee \neg Q_{\Box rp}(y)))$	
$5(\Box(r,\neg(\Box(r,p))))$	$\wedge\ \forall x(Q_{\Box r\neg\Box rp}(x) \vee \forall y(\neg R_r(x,y) \vee \neg Q_{\Box r\neg\Box rp}(y)))$	5o
$Def(\Box(r,\neg(\Box(r,p))))$	$\wedge\ \forall x(Q_{\Box r\neg\Box rp}(x) \leftrightarrow \neg Q_{\neg\Box r\neg\Box rp}(x))$	
	$\wedge\ \forall x(Q_{\Box r\neg\Box rp}(x) \rightarrow \forall y(R_r(x,y) \rightarrow Q_{\neg\Box rp}(y)))$	
	$\wedge\ \forall x(Q_{\neg\Box r\neg\Box rp}(x) \rightarrow \exists y(R_r(x,y) \wedge Q_{\Box rp}(y)$	
□ Instantiation set … { $\Box(r,p)$, $\Box(r,\neg(\Box(r,p)))$}		T
$T(\Box(r,p))$	$\wedge\ \forall x(\neg Q_{\Box rp}(x) \vee Q_p(x))$	
$T(\Box(r,\neg(\Box(r,p))))$	$\wedge\ \forall x(\neg Q_{\Box r\neg\Box rp}(x) \vee Q_{\neg\Box rp}(x))$	

Figure 2.16 <u>The axiomatic translation of a target formula in axiom $K5^2_o$</u>.

The translation for ¬□□□p∧□□p in $K5^2$ and $K5^3$ are shown. The additional formulae marked * arise by considering all the sub-formulae of the new compositional formulae.

For ¬□□□p∧□□p in $K5^2$ the translation is:

∏(¬□□□p∧□□p)

∧ 5^2(□□□p) ∧ 5^2(□□p) ∧5^2(□p)

∧ Def(□□¬□□□p) ∧ Def(□□¬□□p) ∧ Def(□□¬□p)∧ Def(□¬□□□p)* ∧ Def(□¬□□p)* ∧ Def(□¬□p)*

∧ 5^2(□□¬□□□p) ∧ 5^2(□□¬□□p) ∧ 5^2(□□¬□p)∧ 5^2(□¬□□□p)* ∧ 5^2(□¬□□p)* ∧ 5^2(□¬□p)*

For ¬□□□p∧□□p in $K5^3$ the translation is:

∏(¬□□□p∧□□p)

∧ 5^3(□□□p) ∧ 5^3(□□p) ∧5^3(□p) ∧ Def(□□□¬□□□p) ∧ Def(□□□¬□□p) ∧ Def(□□□¬□p)

∧ Def(□□¬□□□p)* ∧ Def(□□¬□□p)* ∧ Def(□□¬□p)*∧ Def(□¬□□□p)* ∧ Def(□¬□□p)* ∧ Def(□¬□p)*

∧ 5^3(□□□¬□□□p) ∧ 5^3(□□□¬□□p) ∧ 5^3(□□□¬□p)

∧ 5^3(□□¬□□□p)* ∧ 5^3(□□¬□□p)* ∧ 5^3(□□¬□p)*∧ 5^3(□¬□□□p)* ∧ 5^3(□¬□□p)* ∧ 5^3(□¬□p)*

Table 2.17 <u>Axiomatic translation with combinations of axioms</u>. (Gathered from [1]). The instantiation sets for axiom combinations are shown, together with whether shortcuts are required or optional. The theorems refer to those described in [1].

Theorem in [1]	Axiom combination	Instantiation set χ_A	Include positive shortcuts
Theorem 5.1	`K4`	$\chi_4 = \chi^{\in}_{\varphi}$	optional
	`KT`	$\chi_T = \chi^{\in}_{\varphi}$	
	`KD`	$\chi_D = \chi^{\in}_{\varphi}$	
	`KB`	$\chi_B = \chi^{\in}_{\varphi}$	
	$\texttt{Kalt}_1$	$\chi_{alt1} = \chi^{\in}_{\varphi}$	
Theorem 5.2	$\texttt{K4}^K$	$\chi_{4K} = \chi^{\in}_{\varphi}$	optional
	$\texttt{Kalt1}^{KK}$	$\chi_{alt1kk} = \chi^{\in}_{\varphi}$	
Theorem 5.3	`KT4 (S4)`	$\chi_T = \chi_4 = \chi^{\in}_{\varphi}$	optional
	`KTB`	$\chi_T = \chi_B = \chi^{\in}_{\varphi}$	
	`KDB`	$\chi_D = \chi_B = \chi^{\in}_{\varphi}$	
	`KD4`	$\chi_D = \chi_4 = \chi^{\in}_{\varphi}$	
Theorem 5.4	`K5`	$\chi_5 = \chi^{5}_{\varphi} = \chi^{\in}_{\varphi} \cup \neg\Box\chi^{\in}_{\varphi}$	required
Theorem 5.5	$\texttt{K5}^K$	$\chi_{5K} = \chi^{5K}_{\varphi} = \chi^{\in}_{\varphi} \cup \Box^{K-1}\neg\Box\chi^{\in}_{\varphi}$	required
Theorem 5.6	`K4B`	$\chi_4 = \chi^{\in}_{\varphi}$ and $\chi_B = \chi^{4}_{\varphi}$	required
Theorem 5.7	`KT4B (S5)`	$\chi_T = \chi_4 = \chi^{\in}_{\varphi}$ and $\chi_B = \chi^{4}_{\varphi}$	required

Table 2.18 <u>Composition of the axiomatic translation of common modal axioms</u>. The compositional subformulae needed for translations incorporating modal axioms are listed. p is an arbitrary modal formula. This table is developed from material in [1]. Some of these compositional subformulae were previously implemented in ml2dfg (mentioned in [1]).

Axiom	Modal representation of axiom	For composition
T_o	$\Box p \rightarrow p$	Nothing
B_o	$\neg\Box\neg\Box p \rightarrow p$	$\Box\neg\Box p$
D_o	$\Box p \rightarrow \neg\Box\neg p$	$\Box\neg p$
4^K_o 4_o 4^2_o 4^3_o	$\Box p \rightarrow \Box k\Box p$ $\Box p \rightarrow \Box\Box p$ $\Box p \rightarrow \Box\Box\Box p$ $\Box p \rightarrow \Box\Box\Box\Box p$	$\Box^K\Box p$ $\Box\Box p$ $\Box\Box\Box p$ $\Box\Box\Box\Box p$
5^K_o 5_o 5^2_o 5^3_o	$\neg\Box k\neg\Box p \rightarrow \Box p$ $\neg\Box\neg\Box p \rightarrow \Box p$ $\neg\Box\Box\neg\Box p \rightarrow \Box p$ $\neg\Box\Box\Box\neg\Box p \rightarrow \Box p$	$\Box^K\neg\Box p$ $\Box\neg\Box p$ $\Box\Box\neg\Box p$ $\Box\Box\Box\neg\Box p$
$alt_1{}^{K1K2}{}_o$ alt_{1o} $alt_1{}^{1,1}{}_o$ $alt_1{}^{1,2}{}_o$ $alt_1{}^{2,1}{}_o$ $alt_1{}^{2,2}{}_o$	$\neg\Box^{K1}\Box p \rightarrow \Box^{K2}\Box\neg p$ $\neg\Box p \rightarrow \Box\neg p$ $\neg\Box\Box p \rightarrow \Box\Box\neg p$ $\neg\Box\Box p \rightarrow \Box\Box\Box\neg p$ $\neg\Box\Box\Box p \rightarrow \Box\Box\neg p$ $\neg\Box\Box\Box p \rightarrow \Box\Box\Box\neg p$	$\Box^{K2}\Box\neg p$ & $\Box^{K1}\Box p$ $\Box\neg p$ $\Box\Box p$ & $\Box\Box\neg p$ $\Box\Box p$ & $\Box\Box\Box\neg p$ $\Box\Box\Box p$ & $\Box\Box\neg p$ $\Box\Box\Box p$ & $\Box\Box\Box\neg p$

Figure 2.19 <u>Derivation of composition of the axiomatic translation of some common modal axioms</u>.

The derivation of compositional formulae for axioms is illustrated with a simple example $\varphi = \Box p$. The general (table 2.18) case can be inferred from each result. The following hold in every case:

$Sf(\varphi) = \{\Box p, p\}$, $\Box\chi = \{\Box p\}$, $\chi^{\in}_{\varphi} = \{p\}$, $\Box\chi^{\in}_{\varphi} = \{\Box p\}$, $\psi \in \{p\}$, $\Box\psi \in \{\Box p\}$

<u>$\varphi = \Box p$ in T_o:</u> $\chi^{T}_{\varphi} = \chi^{\in,T}_{\varphi} = \chi^{\in}_{\varphi} \cup \{ \phi\{p/\psi\} \mid \Box\phi \in \Box\chi^{\in}_{T}, \Box\psi \in Sf(\Box\chi^{\in}_{\varphi}) \}$. Substituting ψ in $\Box\chi^{\varepsilon}_{4}$, for $\neg\Box\psi \vee \psi$ for axiom T. $\neg\Box p \vee p$ $\{\Box p\}$ so $\Box\phi \in \{\Box p\}$ and $\phi \in \{p\}$ and $\chi^{T}_{\varphi} = \{p\} \cup \{p\} = \{p\}$ Finally, $\Box\chi^{T}_{\varphi} = \{\Box p\}$ with no new formulae.	**<u>$\varphi = \Box p$ in D_o:</u>** $\chi^{D}_{\varphi} = \chi^{\in,D}_{\varphi} = \chi^{\in}_{\varphi} \cup \{ \phi\{p/\psi\} \mid \Box\phi \in \Box\chi^{\in}_{D}, \Box\psi \in Sf(\Box\chi^{\in}_{\varphi}) \}$. Substituting ψ in $\Box\chi^{\varepsilon}_{D}$, for $\neg\Box\psi \vee \neg\Box\neg\psi$ for axiom D. $\neg\Box p \vee \neg\Box\neg p$ $\{\Box\neg p, \Box p\}$ so $\Box\phi \in \{\Box\neg p, \Box p\}$ and $\phi \in \{\neg p, p\}$ and $\chi^{T}_{\varphi} = \{p\} \cup \{\neg p, p\} = \{\neg p, p\}$ Finally, $\Box\chi^{T}_{\varphi} = \{\Box\neg p, \Box p\}$ with new formulae of the form $\Box\neg p$.
<u>$\varphi = \Box p$ in B_o:</u> $\chi^{B}_{\varphi} = \chi^{\in,B}_{\varphi} = \chi^{\in}_{\varphi} \cup \{ \phi\{p/\psi\} \mid \Box\phi \in \Box\chi^{\in}_{B}, \Box\psi \in Sf(\Box\chi^{\in}_{\varphi}) \}$. Substituting ψ in $\Box\chi^{\varepsilon}_{B}$, for $\Box\neg\Box\psi \vee \psi$ for axiom B. $\Box\neg\Box p \vee p$ $\{\Box\neg\Box p\}$ so $\Box\phi \in \{\Box\neg\Box p\}$ and $\phi \in \{\neg\Box p\}$ and $\chi^{B}_{\varphi} = \{ p \} \cup \{ \neg\Box p \} = \{ \neg\Box p, p\}$ Finally, $\Box\chi^{B}_{\varphi} = \{\Box\neg\Box p, \Box p\}$ with new formulae of the form $\Box\neg\Box p$.	**<u>$\varphi = \Box p$ in 5_o:</u>** $\chi^{5}_{\varphi} = \chi^{\in,5}_{\varphi} = \chi^{\in}_{\varphi} \cup \{ \phi\{p/\psi\} \mid \Box\phi \in \Box\chi^{\in}_{5}, \Box\psi \in Sf(\Box\chi^{\in}_{\varphi}) \}$. Substituting ψ in $\Box\chi^{\varepsilon}_{5}$, for $\Box\neg\Box\psi \vee \Box\psi$ for axiom 5_o. $\Box\neg\Box p \vee \Box p$ $\{ \Box p, \Box\neg\Box p \}$ so $\Box\phi \in \{\Box p, \Box\neg\Box p\}$ and $\phi \in \{p, \neg\Box p\}$ and $\chi^{5}_{\varphi} = \{p\} \cup \{ p, \neg\Box p \} = \{p, \neg\Box p\}$ Finally, $\Box\chi^{5}_{\varphi} = \{\Box p, \Box\neg\Box p\}$ with new formulae of the form $\Box\neg\Box p$.
<u>$\varphi = \Box p$ in alt_{1o}:</u> $\chi^{alt1}_{\varphi} = \chi^{\in.alt1}_{\varphi} = \chi^{\in}_{\varphi} \cup \{\phi\{p/\psi\} \mid \Box\phi \in \Box\chi^{\in}_{alt1}, \Box\psi \in Sf(\Box\chi^{\in}_{\varphi}) \}$. Substituting ψ in $\Box\chi^{\varepsilon}_{alt1}$, for $\Box\psi \vee \Box\neg\psi$ for axiom alt_1. $\Box p \vee \Box\neg p$ $\{\Box\neg p, \Box p\}$ so $\Box\phi \in \{\Box\neg p, \Box p\}$ and $\phi \in \{\neg p, p\}$ and $\chi^{alt1}_{\varphi} = \{p\} \cup \{\neg p, p\} = \{\neg p, p\}$ Finally, $\Box\chi^{alt1}_{\varphi} = \{\Box\neg p, \Box p\}$ with new formulae of the form $\Box\neg p$.	**<u>$\varphi = \Box p$ in 4_o:</u>** see the text

2.2.4 <u>Shortcuts.</u> [1]

Shortcuts have already been mentioned in formulae 2.2 and 2.2.4. The default definition contains the expression $\forall x(Q_{\psi}(x) \rightarrow \neg Q_{\sim\psi}(x))$, extended by conjugation with $\forall x(Q_{\psi}(x) \leftarrow \neg Q_{\sim\psi}(x))$, to $\forall x(Q_{\psi}(x) \leftrightarrow \neg Q_{\sim\psi}(x))$. The reverse implication here is known as the *positive shortcut* (named in this way because this is a positive formulae when expressed in clausal form). The translation with positive shortcuts included is chosen as the default translation because the positive shortcuts act as *guesses* to the successful proof or refutation, and can result in both shorter proofs, and quicker calculations (although this is not always the case [1, 12 chapter 8]). Positive shortcuts do not add any additional information, as can be seen if the two forward and reverse formulae are considered under resolution. (They are simply a restatement of modus ponens [12]). For the translation of many modal axiom combinations, these shortcuts are in fact required to ensure a complete translation (see table 2.17). In other cases, they are optional, but usually desirable.

2.2.5 Mixed translation modes. [1]

In [1] several mixed modes of translation are considered, in which the axiomatic translation of the target formula, is combined with the instantiation of the axiomatic schema encodings of a subset of the valid modal axioms, while other axioms are represented the correspondence property. These are shown in figure 2.20. Potentially, such mixed mode translations can take advantage of some beneficial properties of both modes of translation. For example, a mixed translation would be beneficial if the correspondence property produced a faster result under resolution than the axiomatic schema encoding for one modal axiom, and the axiomatic schema encoding produced a faster result for the other modal axiom.

Figure 2.20 Mixed mode translations.

Mixed mode translations are listed that have been shown to be sound and complete in [1].

KT4B = S5	=	Corr(T,B) $\wedge \prod^4$ where $\chi_4 = \chi^{\in}_{\varphi}$	(with shortcuts optional)
KDB	=	Corr(D) $\wedge \prod^B$ where $\chi_B = \chi^{\in}_{\varphi}$	(with shortcuts optional)
KD4	=	Corr(D) $\wedge \prod^4$ where $\chi_4 = \chi^{\in}_{\varphi}$	(with shortcuts optional)

2.2.6 Multi-modal target formulae.

In multimodal target formulae, it is possible to distinguish between different instances of the same modal operator. In the notation used in this study, a modality index is attached to each modal operator, and indicates both it's scope and allegiances within the modal formula. Each modality index is associated with a unique accessibility relationship. In the previous uni-modal examples, this modality index has been denoted by r in $\Box$(r,p) and $\Diamond$(r,p), and in the translated formulae, R is the corresponding accessibility relationship. The notation supporting multi-modal operators will be illustrated for example, by the target formula $\Box$(r, ($\neg\Box$(s,p) $\wedge$ $\Box$(t,q))). The basic translation is as seen previously (in figure 2.1) except that the symbols representing the accessibility relations will be different for each case of the box operator. The sub-formulae Sf(φ) are as follows { $\Box$(r, ($\neg\Box$(s,p) $\wedge$ $\Box$(t,q))), $\neg\Box$(s,p) $\wedge$ $\Box$(t,q), $\Box$(s,p), $\Box$(t,q), p, q }, and the translation is shown in figure 2.21. Modality indices have been omitted in previous examples because the cluttering is excessive. However, these indices are very important for multi-modal examples, and must always be included since ambiguity can arise. Hence, Def($\Box$(s,p)) is different from Def($\Box$(t,p)), even though in the truncated notation they both appear to be Def($\Box$p). Differences in the multi-modal translation are particularly important when it comes to applying axioms. An example is seen in figure 2.22, without compositional.

Notice, that the instantiation set for each axiom is effectively *edited* according to the modality index. A further example illustrates this editing and composition in multimodal cases (figures 2.23). The behavior of combinations of axioms in the multimodal case is not defined in [1], and is inferred here without proof. When axioms are applied to different modalities, then they do not interact via the instantiation set; that is, the elements in the instantiation set carry an explicit reference to the modality index. When axioms are referring to the same modality, then their interaction is as already described.

Figure 2.21 <u>Translation of multi-modal target formulae</u>.

The target formula $\square(r, (\neg\square(s,p) \wedge \square(t,q)))$ is translated. R, S, and T are the accessibility relations corresponding to modality indices r, s, t.

$\Pi(\square(r, (\neg\square(s,p) \wedge \square(t,q)))) \quad = \quad \exists x Q_{\square(r,(\neg\square(s,p)\wedge\square(t,q)))}(x)$

$\wedge \quad \forall x(Q_{\square(r,(\neg\square(s,p)\wedge\square(t,q)))}(x) \rightarrow \forall y(R(x,y)\rightarrow Q_{\neg\square(s,p)\wedge\square(t,q)}(y))$ [Def(□(r,(¬□(s,p)∧□(t,q))))]

$\wedge \quad \forall x(Q_{\square(r,(\neg\square(s,p)\wedge\square(t,q)))}(x) \leftrightarrow \neg Q_{\neg\square(\neg\square p\wedge\square q)}(x))$

$\wedge \quad \forall x(Q_{\neg\square(r,(\neg\square(s,p)\wedge\square(t,q)))}(x) \rightarrow \exists y(R(x,y)\wedge Q_{\neg(\neg\square(s,p)\wedge\square(t,q))}(y))$

$\wedge \quad \forall x(Q_{\neg\square(s,p)\wedge\square(t,q)}(x) \rightarrow (Q_{\neg\square(s,p)}(x)\wedge Q_{\square(t,q)}(x))$ [Def(¬□(s,p)∧□(t,q))]

$\wedge \quad \forall x(Q_{\neg\square(s,p)\wedge\square(t,q)}(x) \leftrightarrow \neg Q_{\neg(\neg\square(s,p)\wedge\square(t,q))}(x))$

$\wedge \quad \forall x(Q_{\neg(\neg\square(s,p)\wedge\square(t,q))}(x) \rightarrow (Q_{\square(s,p)}(x)\vee Q_{\neg\square(t,q)}(x))$

$\wedge \quad \forall x(Q_{\square(s,p)}(x) \rightarrow \forall y(S(x,y)\rightarrow Q_p(y))$ [Def(□(s,p))]

$\wedge \quad \forall x(Q_{\square(s,p)}(x) \leftrightarrow \neg Q_{\neg\square(s,p)}(x))$

$\wedge \quad \forall x(Q_{\neg\square(s,p)}(x) \rightarrow \exists y(S(x,y)\wedge Q_{\neg p}(y))$

$\wedge \quad \forall x(Q_{\square(t,q)}(x) \rightarrow \forall y(T(x,y)\rightarrow Q_q(y))$ [Def(□(t,q))]

$\wedge \quad \forall x(Q_{\square(t,q)}(x) \leftrightarrow \neg Q_{\neg\square(t,q)}(x))$

$\wedge \quad \forall x(Q_{\neg\square(t,q)}(x) \rightarrow \exists y(T(x,y)\wedge Q_{\neg q}(y))$

$\wedge \quad \forall x(Q_p(x) \leftrightarrow \neg Q_{\neg p}(x)) \wedge \quad \forall x(Q_q(x) \leftrightarrow \neg Q_{\neg q}(x))$ [from Def(p) & Def(q)]

Figure 2.22 <u>Translation of multi-modal target formulae in axioms T_r, B_s and D_t</u>.

The target formula $\square(r, (\neg\square(s,p) \wedge \square(t,q)))$ is translated in axiom T for r, axiom B for s, and axiom D for t, without composition.

$\Pi(\square(r, (\neg\square(s,p) \wedge \square(t,q))))$

$\wedge \quad \forall x(\neg Q_{\square(r,(\neg\square(s,p)\wedge\square(t,q)))}(x) \vee Q_{\neg\square(s,p)\wedge\square(t,q)}(x))$ T,r(□(r, (¬□(s,p) ∧ □(t,q))))

$\wedge \quad \forall x(\forall y(\neg S(x,y)\vee\neg Q_{\square(s,p)}(y)) \vee Q_p(x))$ B,s(□(s,p)])

$\wedge \quad \forall x(\neg Q_{\square(t,q)}(x) \vee Q_{\neg\square(t,\neg q)}(x))$ D,t(□(t,q)])

$\wedge \quad Def(\square(t,\neg q))$

Figure 2.23 <u>Translation of multi-modal target formulae in with composition</u>.

The translation of $\neg\square(r,\neg\square(a,p))\wedge\neg\square(a,\neg\square(r,\neg p))$ with axiom 5_o applied only to modality index a.

$Def(\neg\square(r,\neg\square(a,p)) \quad \wedge \quad \neg\square(a,\neg\square(r,\neg p))) \quad \wedge \quad Def(p)$

$\wedge \quad Def(\square(r,\neg\square(a,p))) \quad \wedge \quad Def(\square(a,\neg\square(r,\neg p))$

$\wedge \quad Def(\square(a,p)) \quad \wedge \quad Def(\square(r,\neg p))$

$\wedge \quad 5(\square(a,\neg\square(r,\neg p)) \quad \wedge \quad 5(\square(a, \neg\square(a,\neg\square(r,\neg p))) \quad \wedge \quad Def(\square(a,\neg\square(a,\neg\square(r,\neg p)))$

$\wedge \quad 5(\square(a,p)) \quad \wedge \quad 5(\square(a,\neg\square(a,p))) \quad \wedge \quad Def(\square(a,\neg\square(a,p)))$

2.2.7 Bi-modal axioms.

Bi-modal axioms are similar to multimodal formulae. The bi-modal axiom is defined in terms of two box operator species, each identified by an independent modality index. The input target formula will usually have at least two corresponding modality indices. The bi-modal axioms considered in this study are listed in figure 2.2.4. The two modality indices in the bi-modal axiom need to be defined in terms of (or mapped to) the two modality indices in the input problem (potentially both can map to the same modality index). During instantiation of the schema for the bi-modal axiom, these two modality indices are treated differently, as the examples in figure 2.25 illustrate. (Note an alternative notation is used here: □(r,p)) is written [r]p. The change in notation is arbitrary). If an instantiation set {[r]¬[s]p, [r]p, [s]p, [s]¬[r]p, [r]¬[r]p} is considered (there is actually no need to define the original target formula) for modal axioms CR3 and CR2, then there are four possible ways in which the modality indices can be arranged. Naming the modality indices as follows, the "*r*" in the table 2.24 is r_A, and "*s*" is s_A, and the modality indices from the target formula are likewise r_T and s_T, then the four possible mappings of the modality indices in the axiom to the modality indices in the target formula are as listed in table 2.25. The instantiation of the schema clauses for each mapping is also given in table 2.25. It is seen that in effect the first modality index is used to *edit* the instantiation set.

No proof of this material is offered. It represents a logical extension of the axiomatic translation that is suggested in [1]. A derivation is given in the results section.
Note that while multi-modal target formulae and bi-modal axioms are considered in this study, n-ary axioms (referring to more than one modal formula) are not considered. An example would be axiom H (Hintikka corresponding to modal formula ¬□(□p→q)→□(□q→p)).

Figure 2.24 **Schema Encoding for Bi-modal Axioms**. The modal representation of axioms is shown, together with the schema encoding, new terms that need to be Defined during translation, and terms added to the instantiation set during composition. Axioms CR2/CR3 were previously implemented in `ml2dfg` (mentioned in [1]) as axioms A/P.

Axiom	Correspondence Property	Axiomatic Translation		
		Schema encoding	New Term	Composition Term
CR: [r]p → [s]p		$\forall x(\neg Q_{[r]p}(x) \vee Q_{[s]p}(x))$	None	None
CR2: [r]p → [s][r]p	$(R_s(x,y) \wedge R_r(y,z)) \rightarrow R_r(x,z)$	$\forall x(\neg Q_{[r]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[r]p}(y)))$	None	[s][r]p
CR3: [r]p → [r][s]p	$(R_r(x,y) \wedge R_s(y,z)) \rightarrow R_r(x,z)$	$\forall x(\neg Q_{[r]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[s]p}(y)))$	None	[r][s]p

Figure 2.25 **<u>The possible mappings of modality indices from bi-modal target formulae and bi-modal axioms</u>**. (See the text of 2.2.7 for an explanation). The semantics of bimodal axioms was previously implemented in `ml2dfg` [mentioned in 1].

Mapping	**Edited instantiation set**	**Instantiation of schema encoding for CR3**	**Instantiation of schema encoding for CR2**
$r_A \mapsto r_T$ $s_A \mapsto r_T$	[r]p [r]¬[s]p [r]¬[r]p	$\forall x(\neg Q_{[r]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[r]p}(y)))$ $\forall x(\neg Q_{[r]\neg[s]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[r]\neg[s]p}(y)))$ $\forall x(\neg Q_{[r]\neg[r]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[r]\neg[r]p}(y)))$	$\forall x(\neg Q_{[r]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[r]p}(y)))$ $\forall x(\neg Q_{[r]\neg[s]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[r]\neg[s]p}(y)))$ $\forall x(\neg Q_{[r]\neg[r]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[r]\neg[r]p}(y)))$
$r_A \mapsto r_T$ $s_A \mapsto s_T$	[r]p [r]¬[s]p [r]¬[r]p	$\forall x(\neg Q_{[r]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[s]p}(y)))$ $\forall x(\neg Q_{[r]\neg[s]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[s]\neg[s]p}(y)))$ $\forall x(\neg Q_{[r]\neg[r]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[s]\neg[r]p}(y)))$	$\forall x(\neg Q_{[r]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[r]p}(y)))$ $\forall x(\neg Q_{[r]\neg[s]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[r]\neg[s]p}(y)))$ $\forall x(\neg Q_{[r]\neg[r]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[r]\neg[r]p}(y)))$
$r_A \mapsto s_T$ $s_A \mapsto r_T$	[s]p [s]¬[r]p	$\forall x(\neg Q_{[s]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[r]p}(y)))$ $\forall x(\neg Q_{[s]\neg[r]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[r]\neg[r]p}(y)))$	$\forall x(\neg Q_{[s]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[s]p}(y)))$ $\forall x(\neg Q_{[s]\neg[r]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[s]\neg[r]p}(y)))$
$r_A \mapsto s_T$ $s_A \mapsto s_T$	[s]p [s]¬[r]p	$\forall x(\neg Q_{[s]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[s]p}(y)))$ $\forall x(\neg Q_{[s]\neg[r]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[s]\neg[r]p}(y)))$	$\forall x(\neg Q_{[s]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[s]p}(y)))$ $\forall x(\neg Q_{[s]\neg[r]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[s]\neg[r]p}(y)))$

2.3 <u>Resolution.</u>

The authors of [1] go to considerable effort to demonstrate completeness of the axiomatic translation under resolution for the cases described above. Soundness is much more easily proven [1]. As a result it is shown in [1] that for the both axiomatic and mixed modes of translation, that (i) a target formula is (un)satisfiable in the K if and only if the translation of the formulae is (un)satisfiable in first-order logic, and (ii) a target formula is (un)satisfiable in the sequence of modal axioms $K\alpha_1 \ldots \alpha_n$, if and only if the translation of the formulae in this sequence of modal axioms is (un)satisfiable in first-order logic. This is the basis on which the satisfiability of modal target formulae is determined in extended-SPASS. (See section 7 for examples).

There are other points that are worth noting. It is shown in [1] that the axiomatic translation of modal target formulae with different modal axioms, yields formulae that are known to be *decidable* under first-order ordered resolution [37]. This is because only formulae composed of fragments known to be decidable can be formed by the translation. (These decidable fragments are members of the classes of formulae known as guarded fragments (GF^2, [1, 23]) or DL* fragments [1, 22, 34]). This is an important result. There is no direct means by which formulae with 'triangular' properties, of the types embodied in the modal axioms 4, 5, and alt_1 can be always expressed in these (or other) decidable fragments. This then is the major advantage of the axiomatic translation. It is expected that many problems that are impossible to solve in axioms 4, 5 and alt_1, will be soluble under axiomatic translation. Indeed this is the finding in [1] and in the results section 7.