2. The Axiomatic Translation of Modal Logic [1]: *DRAFT* @K.J.Smith, 2008. http://www.KJSmith.net

This entire section is based upon, but heavily modified from, the text of [1]. The material has been re-ordered to better follow a narrative description, and many derivations and examples have been added throughout, to support the description. Some material is extended from results suggested in [1].

2.0. Axiomatic translation of the input modal problem.

If the complications are ignored, then the axiomatic translation is quite simple. A modal formula φ is processed (under the function $\prod$) as follows. All modal (that is, box) symbols are eliminated by this translation, giving rise to a formula in first-order logic.

$$\prod(\varphi) = \exists x\, Q_\varphi(x) \wedge \bigwedge\{Def(\psi) \mid \psi \in Sf(\varphi)\}$$ [formula 2.1, see clause 1, definition 4.1 in [1]]

here $Sf(\varphi)$ denotes the set of all *sub-formulae* of φ, and Q_φ is a predicate symbol uniquely associated with the modal formula φ. The sub-formula are each processed as follows for the definition of $Def(\psi)$, and the results added to the others by repeated conjugation ($\wedge$).

$Def(\psi) =$	$\forall x(Q_\psi(x) \rightarrow \pi(\psi, x))$	
$\wedge$	$\forall x(Q_\psi(x) \leftrightarrow \neg Q_{\sim\psi}(x))$	[from $\forall x(Q_\psi(x)\rightarrow\neg Q_{\sim\psi}(x))\wedge\forall x(Q_\psi(x)\leftarrow\neg Q_{\sim\psi}(x))$]
$\wedge$	$\forall x(Q_{\sim\psi}(x) \rightarrow \pi(\sim\psi, x))$	[formula 2.2, see section 3 in [1]]

Note this definition includes the reverse implication for the second formula (see section 2.1.4)

Note that complement ($\sim$) is defined so $\sim\phi=\neg\phi$ and $\sim\neg\phi=\phi$.

where $\pi(\bot, x)$	=	$\bot$			
$\pi(p, x)$	=	$\top$	$\pi(\neg p, x)$	=	$\neg Q_p(x)$
$\pi(\psi\wedge\phi, x)$	=	$Q_\psi(x) \wedge Q_\phi(x)$	$\pi(\neg(\psi\wedge\phi), x)$	=	$Q_{\sim\psi}(x) \vee Q_{\sim\phi}(x)$
$\pi(\Box\psi, x)$	=	$\forall y(R(x, y) \rightarrow Q_\psi(y))$			
$\pi(\neg\Box\psi, x)$	=	$\exists y(R(x, y) \wedge Q_{\sim\psi}(y))$			[formulae 2.3, see section 3 in [1]]

An obvious extension is to consider the non-binary case of conjunction, for example in the ternary case

$\pi(\psi\wedge\phi\wedge\varphi, x)$	=	$Q_\psi(x) \wedge Q_\phi(x) \wedge Q_\varphi(x)$	
$\pi(\neg(\psi\wedge\phi\wedge\varphi), x)$	=	$Q_{\sim\psi}(x) \vee Q_{\sim\phi}(x) \vee Q_{\sim\varphi}(x)$	[formulae 2.3.1]

It is easy to show that under resolution, the clauses for example from $Def(\psi\wedge\phi\wedge\varphi)$, are identical in nature to those arising from the binary cases $Def((\psi\wedge\phi)\wedge\varphi)$, $Def(\psi\wedge(\phi\wedge\varphi))$, and $Def(\phi\wedge(\varphi\wedge\psi))$. A similar result can be inferred for any number of conjugated formulae (data not presented).

Note that the translation of the modal formula (φ) has a single iteration, unlike the classical translation that applies the formulae 1.? inductively (in multiple iterations) until all modal symbols are eliminated. Here the iteration is replaced by consideration the sub-formulae of φ. The resulting definitions are linked, one to another, by the predicate symbols that are generated (each uniquely associated with a particular sub-formulae of the input modal formula). It should be noted that the translation is only defined for formulae with the following operators, $\wedge$, $\neg$, and $\Box$ (with $\bot$). Any modal formula to be translated must first be modified, by simple transformations (see section ?), in order to exclude other operators. This simplifies the axiomatic translation.

An example of *manual* manipulation of a modal formula under axiomatic translation follows, taking as an example the modal formula $\varphi = \Box((\neg\Box(r,p)) \wedge \Box(r,q))$. First, it is necessary to define the sub-formulae of the modal formula φ, as marked by an arrow below. Essentially every possible sub-formula that can be defined within the parent formula is identified. For simplicity the modality index, r has been dropped in the following example (possible only because the example is uni-modal). R is the binary predicate symbol associated with the modality index r.

↓		↓	↓	↓	↓	↓	↓		
$\Box$	(	$(\neg$	$\Box$	p)	$\wedge$	$\Box$	q	)	

Listing the sub-formulae gives

1.	$\Box((\neg\Box p)\wedge\Box q)$	5.	p
2.	$\neg\Box p\wedge\Box q$	6.	$\Box q$
3.	$\neg\Box p$	7.	q
4.	$\Box p$		

This set of sub-formulae is known as the default *instantiation set* for this modal formula.

Proceeding to the translation, hence,

$\prod(\Box((\neg\Box p) \wedge \Box q))$	=	$\exists x Q_{\Box(\neg\Box p\wedge\Box q)}(x)$	
	$\wedge$	$\forall x(Q_{\Box(\neg\Box p\wedge\Box q)}(x) \rightarrow \pi(\Box((\neg\Box p)\wedge\Box q), x))$	[from Def($\Box((\neg\Box p)\wedge\Box q)$) (Sf1)]
	$\wedge$	$\forall x(Q_{\Box(\neg\Box p\wedge\Box q)}(x) \leftrightarrow \neg Q_{\sim\Box(\neg\Box p\wedge\Box q)}(x))$	
	$\wedge$	$\forall x(Q_{\sim\Box(\neg\Box p\wedge\Box q)}(x) \rightarrow \pi(\sim\Box((\neg\Box p)\wedge\Box q), x))$	
	$\wedge$	$\forall x(Q_{\neg\Box p\wedge\Box q}(x) \rightarrow \pi((\neg\Box p)\wedge\Box q, x))$	[from Def($(\neg\Box p)\wedge\Box q$) (Sf2)]
	$\wedge$	$\forall x(Q_{\neg\Box p\wedge\Box q}(x) \leftrightarrow \neg Q_{\sim(\neg\Box p\wedge\Box q)}(x))$	
	$\wedge$	$\forall x(Q_{\sim(\neg\Box p\wedge\Box q)}(x) \rightarrow \pi(\sim((\neg\Box p)\wedge\Box q), x))$	
	$\wedge$	$\forall x(Q_{\neg\Box p}(x) \rightarrow \pi(\neg\Box p, x))$	[from Def($\neg\Box p$) (Sf3)]
	$\wedge$	$\forall x(Q_{\neg\Box p}(x) \leftrightarrow \neg Q_{\sim\neg\Box p}(x))$	
	$\wedge$	$\forall x(Q_{\sim\neg\Box p}(x) \rightarrow \pi(\sim\neg\Box p, x))$	
	$\wedge$	$\forall x(Q_{\Box p}(x) \rightarrow \pi(\Box p, x))$	[from Def($\Box p$) (Sf4)]
	$\wedge$	$\forall x(Q_{\Box p}(x) \leftrightarrow \neg Q_{\sim\Box p}(x))$	
	$\wedge$	$\forall x(Q_{\sim\Box p}(x) \rightarrow \pi(\sim\Box p, x))$	
	$\wedge$	$\forall x(Q_{\Box q}(x) \rightarrow \pi(\Box q, x))$	[from Def($\Box q$) (Sf6)]
	$\wedge$	$\forall x(Q_{\Box q}(x) \leftrightarrow \neg Q_{\sim\Box q}(x))$	
	$\wedge$	$\forall x(Q_{\sim\Box q}(x) \rightarrow \pi(\sim\Box q, x))$	
	$\wedge$	$\forall x(Q_p(x) \rightarrow \pi(p, x))$	[from Def(p) (Sf5)]
	$\wedge$	$\forall x(Q_p(x) \leftrightarrow \neg Q_{\sim p}(x))$	
	$\wedge$	$\forall x(Q_{\sim p}(x) \rightarrow \pi(\sim p, x))$	
	$\wedge$	$\forall x(Q_q(x) \rightarrow \pi(q, x))$	[from Def(q) (Sf7)]
	$\wedge$	$\forall x(Q_q(x) \leftrightarrow \neg Q_{\sim q}(x))$	
	$\wedge$	$\forall x(Q_{\sim q}(x) \rightarrow \pi(\sim q, x))$	
$\prod(\Box((\neg\Box p) \wedge \Box q))$	=	$\exists x Q_{\Box(\neg\Box p\wedge\Box q)}(x)$	
	$\wedge$	$\forall x(Q_{\Box(\neg\Box p\wedge\Box q)}(x) \rightarrow \forall y(R(x,y)\rightarrow Q_{\neg\Box p\wedge\Box q}(y))$	[from Sf1)]
	$\wedge$	$\forall x(Q_{\Box(\neg\Box p\wedge\Box q)}(x) \leftrightarrow \neg Q_{\neg\Box(\neg\Box p\wedge\Box q)}(x))$	[$Q_{\sim\Box(\neg\Box p\wedge\Box q)}(x)= Q_{\neg\Box(\neg\Box p\wedge\Box q)}(x)$]
	$\wedge$	$\forall x(Q_{\neg\Box(\neg\Box p\wedge\Box q)}(x) \rightarrow \exists y(R(x,y)\wedge Q_{\neg(\neg\Box p\wedge\Box q)}(y))$	[$\sim\Box(\neg\Box p\wedge\Box q)=\neg\Box(\neg\Box p\wedge\Box q)$]
	$\wedge$	$\forall x(Q_{\neg\Box p\wedge\Box q}(x) \rightarrow (Q_{\neg\Box p}(x)\wedge Q_{\Box q}(x))$	[from Sf2]
	$\wedge$	$\forall x(Q_{\neg\Box p\wedge\Box q}(x) \leftrightarrow \neg Q_{\neg(\neg\Box p\wedge\Box q)}(x))$	
	$\wedge$	$\forall x(Q_{\neg(\neg\Box p\wedge\Box q)}(x) \rightarrow (Q_{\Box p}(x)\vee Q_{\neg\Box q}(x))$	[$\sim(\neg\Box p\wedge\Box q)=\neg(\neg\Box p\wedge\Box q)$; $Q_{\sim(\neg\Box p\wedge\Box q)}(x)= Q_{\neg(\neg\Box p\wedge\Box q)}(x)$; $Q_{\neg\neg\Box p}(x)=Q_{\Box p}(x)$]
	$\wedge$	$\forall x(Q_{\neg\Box p}(x) \rightarrow \exists y(R(x,y)\wedge Q_{\neg p}(y))$	[from Sf3]
	$\wedge$	$\forall x(Q_{\neg\Box p}(x) \leftrightarrow \neg Q_{\Box p}(x))$	[$\sim\neg\Box p = \neg\neg\Box p = \Box p$]
	$\wedge$	$\forall x(Q_{\Box p}(x) \rightarrow \forall y(R(x,y)\rightarrow Q_p(y))$	[$\sim\neg\Box p = \neg\neg\Box p = \Box p$]
	$\wedge$	$\forall x(Q_{\Box p}(x) \rightarrow \forall y(R(x,y)\rightarrow Q_p(y))$	[from Sf4]
	$\wedge$	$\forall x(Q_{\Box p}(x) \leftrightarrow \neg Q_{\neg\Box p}(x))$	[$Q_{\sim\Box}p = Q_{\neg\Box}p$]
	$\wedge$	$\forall x(Q_{\neg\Box p}(x) \rightarrow \exists y(R(x,y)\wedge Q_{\neg p}(y))$	[$Q_{\sim p}(x)= Q_{\neg p}(x)$; $Q_{\sim\Box p}(x) = Q_{\neg\Box p}(x)$]
	$\wedge$	$\forall x(Q_{\Box q}(x) \rightarrow \forall y(R(x,y)\rightarrow Q_q(y))$	[from Sf6]
	$\wedge$	$\forall x(Q_{\Box q}(x) \leftrightarrow \neg Q_{\neg\Box q}(x))$	[$Q_{\sim\Box q}(x) = Q_{\neg\Box q}(x)$]
	$\wedge$	$\forall x(Q_{\neg\Box q}(x) \rightarrow \exists y(R(x,y)\wedge Q_{\neg q}(y))$	[$Q_{\sim\Box q}(x) = Q_{\neg\Box q}(x)$; $Q_{\sim q}(y) = Q_{\neg q}(y)$]
	$\wedge$	$\forall x(Q_p(x) \rightarrow \top)$	[from Sf5]
	$\wedge$	$\forall x(Q_p(x) \leftrightarrow \neg Q_{\neg p}(x))$	[$Q_{\sim p}(x) = Q_{\neg p}(x)$]
	$\wedge$	$\forall x(Q_{\neg p}(x) \rightarrow \neg Q_p(x))$	[$Q_{\sim p}(x) = Q_{\neg p}(x)$]
	$\wedge$	$\forall x(Q_q(x) \rightarrow \top)$	[from Sf7]
	$\wedge$	$\forall x(Q_q(x) \leftrightarrow \neg Q_{\neg q}(x))$	[$Q_{\sim q}(x) = Q_{\neg q}(x)$]
	$\wedge$	$\forall x(Q_{\neg q}(x) \rightarrow \neg Q_q(x))$	[$Q_{\sim q}(x) = Q_{\neg q}(x)$]

It is clear that while the transformations in the axiomatic translation are simple, they give rise to a *great many* new sub-formulae. It is worth noting that it can be shown [1] that the instantiation set and the translation (as described above, and with the extensions for incorporation of modal axioms described in later sections) can be calculated in linear time. Fortunately, there are some further modifications that lead to a considerable simplification. (These simplifications are introduced into the *formal* definition of axiomatic translation in clause 1 of definition 4.1 in [1] – here the text is a more informal description supported by examples).

First if p is a predicate, then Def(p) can be simplified.

Def(p) =	$\forall x(Q_p(x) \rightarrow \pi(p, x))$	$\wedge$	$\forall x(Q_{\neg p}(x) \rightarrow \pi(\sim p, x))$		
$\wedge$	$\forall x(Q_p(x) \rightarrow \neg Q_{\neg p}(x))$	=	$\forall x(Q_p(x) \rightarrow \top)$	[reduces to $\top$]
$\wedge$	$\forall x(Q_p(x) \leftarrow \neg Q_{\neg p}(x))$	$\wedge$	$\forall x(Q_p(x) \rightarrow \neg Q_{\neg p}(x))$	

$$\begin{aligned}
\wedge\quad & \forall x(Q_p(x) \leftarrow \neg Q_{\neg p}(x)) \\
\wedge\quad & \forall x(Q_{\neg p}(x) \rightarrow \neg Q_p(x)) \\
& \quad [Q_{\sim p}(x)\rightarrow\neg Q_p(x) \equiv Q_p(x)\rightarrow\neg Q_{\sim p}(x)] \\
=\quad & \forall x(Q_p(x) \rightarrow \neg Q_{\neg p}(x))
\end{aligned}$$

$$\begin{aligned}
\wedge\quad & \forall x(Q_p(x) \leftarrow \neg Q_{\neg p}(x)) \\
Def(p) =\quad & \forall x(Q_p(x) \rightarrow \neg Q_{\neg p}(x)) \\
\wedge\quad & \forall x(Q_p(x) \leftarrow \neg Q_{\neg p}(x))
\end{aligned}$$

Hence in general for predicate p, the formula 2.2 can be simplified to

$$Def(\psi) = \quad \forall x(Q_\psi(x) \leftrightarrow \neg Q_{\sim\psi}(x)) \qquad [\text{from } \forall x(Q_\psi(x)\rightarrow\neg Q_{\sim\psi}(x))\wedge\forall x(Q_\psi(x)\leftarrow \neg Q_{\sim\psi}(x))]$$

[formula 2.3.2]

Again, this definition includes the reverse implication for the second formula (see section 2.1.4).

Second, if Def(¬ψ) for modal formulae is considered, where the enclosing symbol is 'not' (¬), then it can be shown that Def(¬ψ) and Def(ψ) are equivalent. Since the sub-formulae of ¬ψ must include ψ, inclusion of sub-formulae with an enclosing ¬ symbol in the instantiation set is redundant. The proof involves comparison of non-negated and negated standardized modal formulae, considering all three cases that might occur.

Case 1. For a modal formula ϕ:

$$\begin{aligned}
Def(\phi) =\quad & \forall x(Q_\phi(x) \rightarrow \neg Q_{\neg\phi}(x)) \\
\wedge\quad & \forall x(Q_\phi(x) \leftarrow \neg Q_{\neg\phi}(x))
\end{aligned}$$

Consider ¬ϕ

$$\begin{aligned}
Def(\neg\phi) =\quad & \forall x(Q_{\neg\phi}(x) \rightarrow \pi(\neg\phi, x)) \\
\wedge\quad & \forall x(Q_{\neg\phi}(x) \rightarrow \neg Q_{\sim\neg\phi}(x)) \\
\wedge\quad & \forall x(Q_{\neg p}(x) \leftarrow \neg Q_{\sim\neg p}(x)) \\
\wedge\quad & \forall x(Q_{\sim\neg p}(x) \rightarrow \pi(\sim\neg p, x)) \\
=\quad & \forall x(Q_{\neg p}(x) \rightarrow \pi(\neg p, x)) \\
\wedge\quad & \forall x(Q_{\neg p}(x) \rightarrow \neg Q_p(x)) \\
\wedge\quad & \forall x(Q_{\neg p}(x) \leftarrow \neg Q_p(x)) \\
\wedge\quad & \forall x(Q_p(x) \rightarrow \pi(p, x)) \\
=\quad & \forall x(Q_{\neg\phi}(x) \rightarrow \neg Q_\phi(x)) \\
\wedge\quad & \forall x(Q_{\neg\phi}(x) \rightarrow \neg Q_\phi(x)) \\
\wedge\quad & \forall x(Q_{\neg\phi}(x) \leftarrow \neg Q_\phi(x)) \\
\wedge\quad & \forall x(Q_\phi(x) \rightarrow \top) \quad [\text{reduces to } \top] \\
=\quad & \forall x(Q_{\neg\phi}(x) \rightarrow \neg Q_\phi(x)) \\
\wedge\quad & \forall x(Q_{\neg\phi}(x) \leftarrow \neg Q_\phi(x)) \\
=\quad & \forall x(Q_\phi(x) \rightarrow \neg Q_{\neg\phi}(x)) \\
\wedge\quad & \forall x(Q_\phi(x) \leftarrow \neg Q_{\neg\phi}(x)) \\
=\quad & Def(\phi)
\end{aligned}$$

Case 2. Consider modal formula ϕ

$$\begin{aligned}
Def(\Box\phi) =\quad & \forall x\ (Q_{\Box\phi}(x) \rightarrow \pi(\Box\phi, x)) \\
\wedge\quad & \forall x\ (Q_{\Box\phi}(x) \rightarrow \neg Q_{\sim\Box\phi}(x)) \\
\wedge\quad & \forall x\ (Q_{\Box\phi}(x) \leftarrow \neg Q_{\sim\Box\phi}(x)) \\
\wedge\quad & \forall x\ (Q_{\sim\Box\phi}(x) \rightarrow \pi(\sim\Box\phi, x)) \\
=\quad & \forall x\ (Q_{\Box\phi}(x) \rightarrow \pi(\Box\phi, x)) \\
\wedge\quad & \forall x\ (Q_{\Box\phi}(x) \rightarrow \neg Q_{\neg\Box\phi}(x)) \\
\wedge\quad & \forall x\ (Q_{\Box\phi}(x) \leftarrow \neg Q_{\neg\Box\phi}(x)) \\
\wedge\quad & \forall x\ (Q_{\neg\Box\phi}(x) \rightarrow \pi(\neg\Box\phi, x)) \\
=\quad & \forall x\ (Q_{\Box\phi}(x) \rightarrow \forall y(R(x, y) \rightarrow Q_\phi(y))) \\
\wedge\quad & \forall x\ (Q_{\Box\phi}(x) \rightarrow \neg Q_{\neg\Box\phi}(x)) \\
\wedge\quad & \forall x\ (Q_{\Box\phi}(x) \leftarrow \neg Q_{\neg\Box\phi}(x)) \\
\wedge\quad & \forall x\ (Q_{\neg\Box\phi}(x) \rightarrow \exists y(R(x, y) \wedge Q_{\neg\phi}(y))) \\
=\quad & \forall x\ (Q_{\Box\phi}(x) \rightarrow \forall y(R(x, y) \rightarrow Q_\phi(y))) \\
\wedge\quad & \forall x\ (Q_{\Box\phi}(x) \rightarrow \neg Q_{\neg\Box\phi}(x)) \\
\wedge\quad & \forall x\ (Q_{\Box\phi}(x) \leftarrow \neg Q_{\neg\Box\phi}(x)) \\
\wedge\quad & \forall x\ (Q_{\neg\Box\phi}(x) \rightarrow \exists y(R(x, y) \wedge Q_{\neg\phi}(y)))
\end{aligned}$$

Consider

$$\begin{aligned}
Def(\neg\Box\phi) =\quad & \forall x\ (Q_{\neg\Box\phi}(x) \rightarrow \pi(\neg\Box\phi, x)) \\
\wedge\quad & \forall x\ (Q_{\neg\Box\phi}(x) \rightarrow \neg Q_{\sim\neg\Box\phi}(x)) \\
\wedge\quad & \forall x\ (Q_{\neg\Box\phi}(x) \leftarrow \neg Q_{\sim\neg\Box\phi}(x)) \\
\wedge\quad & \forall x\ (Q_{\sim\neg\Box\phi}(x) \rightarrow \pi(\sim\neg\Box\phi, x)) \\
=\quad & \forall x\ (Q_{\neg\Box\phi}(x) \rightarrow \pi(\neg\Box\phi, x)) \\
\wedge\quad & \forall x\ (Q_{\neg\Box\phi}(x) \rightarrow \neg Q_{\Box\phi}(x)) \\
\wedge\quad & \forall x\ (Q_{\neg\Box\phi}(x) \leftarrow \neg Q_{\Box\phi}(x)) \\
\wedge\quad & \forall x\ (Q_{\Box\phi}(x) \rightarrow \pi(\Box\phi, x)) \\
=\quad & Def(\Box\phi)
\end{aligned}$$

Case 3. Consider modal formulae ϕ and ψ

$$\begin{aligned}
Def(\phi\wedge\psi) =\quad & \forall x(Q_{\phi\wedge\psi}(x) \rightarrow \pi(\phi\wedge\psi, x)) \\
\wedge\quad & \forall x(Q_{\phi\wedge\psi}(x) \rightarrow \neg Q_{\sim(\phi\wedge\psi)}(x)) \\
\wedge\quad & \forall x(Q_{\phi\wedge\psi}(x) \leftarrow \neg Q_{\sim(\phi\wedge\psi)}(x)) \\
\wedge\quad & \forall x(Q_{\sim(\phi\wedge\psi)}(x) \rightarrow \pi(\sim(\phi\wedge\psi), x)) \\
=\quad & \forall x(Q_{\phi\wedge\psi}(x) \rightarrow Q_\phi(x) \wedge Q_\psi(x)) \\
\wedge\quad & \forall x(Q_{\phi\wedge\psi}(x) \rightarrow \neg Q_{\neg(\phi\wedge\psi)}(x)) \\
\wedge\quad & \forall x(Q_{\phi\wedge\psi}(x) \leftarrow \neg Q_{\neg(\phi\wedge\psi)}(x)) \\
\wedge\quad & \forall x(Q_{\neg(\phi\wedge\psi)}(x) \rightarrow Q_{\neg\phi}(x) \vee Q_{\neg\psi}(x))
\end{aligned}$$

Consider

$$\begin{aligned}
Def(\neg(\psi\wedge\phi)) =\quad & \forall x(Q_{\neg(\phi\wedge\psi)}(x) \rightarrow \pi(\neg(\phi\wedge\psi), x)) \\
\wedge\quad & \forall x(Q_{\neg(\phi\wedge\psi)}(x) \rightarrow \neg Q_{\sim\neg(\phi\wedge\psi)}(x)) \\
\wedge\quad & \forall x(Q_{\neg(\phi\wedge\psi)}(x) \leftarrow \neg Q_{\sim\neg(\phi\wedge\psi)}(x)) \\
\wedge\quad & \forall x(Q_{\sim\neg(\phi\wedge\psi)}(x) \rightarrow \pi(\sim\neg(\phi\wedge\psi), x)) \\
=\quad & \forall x(Q_{\neg(\phi\wedge\psi)}(x) \rightarrow \pi(\neg(\phi\wedge\psi), x)) \\
\wedge\quad & \forall x(Q_{\neg(\phi\wedge\psi)}(x) \rightarrow \neg Q_{(\phi\wedge\psi)}(x)) \\
\wedge\quad & \forall x(Q_{\neg(\phi\wedge\psi)}(x) \leftarrow \neg Q_{(\phi\wedge\psi)}(x)) \\
\wedge\quad & \forall x(Q_{(\phi\wedge\psi)}(x) \rightarrow \pi((\phi\wedge\psi), x))
\end{aligned}$$

These equations indicate that Def(ψ) formulae involving negated symbols (¬p, ¬□p, or ¬(p∧q)) can be omitted from the axiomatic translation, since the same formulae will be found in the non-negated sub-formulae (p, □p, or (p∧q)).

Using just these two modifications it is possible to reduce the instantiation set by removing ¬□p and then the example above reduces to

$$\begin{aligned}
\prod(\Box(\neg\Box p\wedge\Box q)) =\quad & \exists x Q_{\Box(\neg\Box p\wedge\Box q)}(x) \\
\wedge\quad & \forall x(Q_{\Box(\neg\Box p\wedge\Box q)}(x) \rightarrow \forall y(R(x,y)\rightarrow Q_{\neg\Box p\wedge\Box q}(y))) && [\text{from(1)}] \\
\wedge\quad & \forall x(Q_{\Box(\neg\Box p\wedge\Box q)}(x) \leftrightarrow \neg Q_{\neg\Box(\neg\Box p\wedge\Box q)}(x)) \\
\wedge\quad & \forall x(Q_{\neg\Box(\neg\Box p\wedge\Box q)}(x) \rightarrow \exists y(R(x,y)\wedge Q_{\neg(\neg\Box p\wedge\Box q)}(y))) \\
\wedge\quad & \forall x(Q_{\neg\Box p\wedge\Box q}(x) \rightarrow (Q_{\neg\Box p}(x)\wedge Q_{\Box q}(x))) && [\text{from(2)}] \\
\wedge\quad & \forall x(Q_{\neg\Box p\wedge\Box q}(x) \leftrightarrow \neg Q_{\neg(\neg\Box p\wedge\Box q)}(x)) \\
\wedge\quad & \forall x(Q_{\neg(\neg\Box p\wedge\Box q)}(x) \rightarrow (Q_{\Box p}(x)\vee Q_{\neg\Box q}(x))) \\
\wedge\quad & \forall x(Q_{\Box p}(x) \rightarrow \forall y(R(x,y)\rightarrow Q_p(y))) && [\text{from(4)}] \\
\wedge\quad & \forall x(Q_{\Box p}(x) \leftrightarrow \neg Q_{\neg\Box p}(x)) \\
\wedge\quad & \forall x(Q_{\neg\Box p}(x) \rightarrow \exists y(R(x,y)\wedge Q_{\neg p}(y))) \\
\wedge\quad & \forall x(Q_{\Box q}(x) \rightarrow \forall y(R(x,y)\rightarrow Q_q(y))) && [\text{from(6)}] \\
\wedge\quad & \forall x(Q_{\Box q}(x) \leftrightarrow \neg Q_{\neg\Box q}(x)) \\
\wedge\quad & \forall x(Q_{\neg\Box q}(x) \rightarrow \exists y(R(x,y)\wedge Q_{\neg q}(y))) \\
\wedge\quad & \forall x(Q_p(x) \leftrightarrow \neg Q_{\neg p}(x)) && [\text{from(5)}] \\
\wedge\quad & \forall x(Q_q(x) \leftrightarrow \neg Q_{\neg q}(x)) && [\text{from(7)}]
\end{aligned}$$

It is clear that the number of terns submitted to resolution will be greatly reduced, and this is likely to yield a faster and shorter proof.

Although it is not relevant to the previous example, a further obvious modification arises from the translation of conjuncted formulae. In formulae 2.2

$$Def(\psi\wedge\phi) = \forall x(Q_{\psi\wedge\phi}(x) \rightarrow \pi(\psi\wedge\phi, x)) \wedge \forall x(Q_{\psi\wedge\phi}(x) \leftrightarrow \neg Q_{\neg(\psi\wedge\phi)}(x)) \wedge \forall x(Q_{\neg(\psi\wedge\phi)}(x) \rightarrow \pi(\neg(\psi\wedge\phi), x))$$

but $$Def(\phi\wedge\psi) = \forall x(Q_{\phi\wedge\psi}(x) \rightarrow \pi(\phi\wedge\psi, x)) \wedge \forall x(Q_{\phi\wedge\psi}(x) \leftrightarrow \neg Q_{\neg(\phi\wedge\psi)}(x)) \wedge \forall x(Q_{\neg(\phi\wedge\psi)}(x) \rightarrow \pi(\neg(\phi\wedge\psi), x))$$

The new predicate symbols introduced are different, for example $Q_{\psi\wedge\phi}$ and $Q_{\phi\wedge\psi}$, but clearly these formulae should give rise to the same translation. In this simple example, there is little significance, but in other examples, duplication of formulae could occur, which will increase the size of the task presented to the SPASS resolution prover. For example, duplication would occur in translation of the sub-formulae of (ψ∧ϕ) →(ϕ∧ψ). The solution is to sort the conjugated formulae – the arguments in the list of an ∧ operator. The sorting function is unimportant, as long as it always returns the same result for a given series of formulae. Clearly sorting should also be applied to conjugated formulae in a non-binary example (e.g. Def(ψ∧ϕ∧φ)) and in the case of negated conjugation (e.g. Def¬(ψ∧ϕ∧φ)).

2.1. Inclusion of modal axioms in the axiomatic translation of modal logic:

As already discussed, modal axioms are often formulated as additional constraints on the accessibility relations of Kripke frames in modal logic. The axioms are theorems of these new restricted systems.

2.1.1. The incorporation of the correspondence properties of common modal axioms:

The classical translations (correspondence properties) of several common axioms of modal logic are given in table 2.1. Different ways in which they can be derived has been sketched in section 1.3. Incorporation of these correspondence properties into the translation procedure given above is simple. The classical translation from the table 2.1 is simply added by repeated conjugation (∧) to the list of translated formulae. In multimodal formula, it is important that the correct accessibility relation (R) in table 2.1 is used.

For example, □((¬□(r,p)) ∧ □(r,q)) in KB_c is simply

$$\begin{aligned}
& \prod(\ \Box((\neg\Box(r,p))\wedge\Box(r,q))\ ) && [\text{as defined above in section 3}] \\
\wedge\quad & \forall xy(R(x,y) \rightarrow R(y,x)) && [\text{written in shorthand as } Corr(B)]
\end{aligned}$$

Table 2.1: Correspondence properties of common modal axioms (Modified from [1], see also section 1.?)

Axiom	Correspondence Property (*Corr*)
T reflexive	$\forall x(R(x,x))$
D serial	$\forall x\exists y(R(x,y))$
B symmetry	$\forall xy(R(x,y) \rightarrow R(y,x))$
4 transitive	$\forall xyz((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$
5 Euclidean	$\forall xyz(R(x,y) \wedge R(x,z) \rightarrow R(y,z))$
alt_1 functional	$\forall xyz(R(x,y) \wedge R(x,z) \rightarrow (y\approx z)\ )$
4^κ 4^2 4^3	$\forall xy(R^{\kappa+1}(x,y) \rightarrow R(x,y))$ $\forall xuyz(\ (R(x,u) \wedge R(u,y) \wedge R(y,z)\ ) \rightarrow R(x,z)\ )$ $\forall xuvyz(\ (R(x,u) \wedge R(u,v) \wedge R(v,y) \wedge R(y,z)\ ) \rightarrow R(x,z)\ )$
5^κ 5^2 5^3	$\forall xyz(R^\kappa(x,y) \wedge R(x,z) \rightarrow R(y,z))$ $\forall xuyz(\ (R(x,u) \wedge R(u,y) \wedge R(x,z)\ ) \rightarrow R(y,z))$ $\forall xuvyz(\ (R(x,u) \wedge R(u,v) \wedge R(v,y) \wedge R(x,z)\ ) \rightarrow R(y,z))$

$alt_1^{\kappa 1,\kappa 2}$	$\forall xyz(R^{\kappa 1+1}(x,y) \wedge R^{\kappa 2+1}(x,z) \rightarrow (y \approx z))$
$alt_1^{1,1}$	$\forall xyz((\exists u(R(x,u) \wedge R(u,y)) \wedge \exists v(R(x,v) \wedge R(v,z))) \rightarrow (y \approx z))$
$alt_1^{1,2}$	$\forall xyz((\exists u(R(x,u) \wedge R(u,y)) \wedge \exists vw(R(x,v) \wedge R(v,w) \wedge R(w,z))) \rightarrow (y \approx z))$
$alt_1^{2,1}$	$\forall xyz((\exists uw(R(x,u) \wedge R(u,w) \wedge R(w,y)) \wedge \exists v(R(x,v) \wedge R(v,z))) \rightarrow (y \approx z))$
$alt_1^{2,2}$	$\forall xyz((\exists uw(R(x,u) \wedge R(u,w) \wedge R(w,y)) \wedge \exists vg(R(x,v) \wedge R(v,g) \wedge R(g,z))) \rightarrow (y \approx z))$

2.1.2. <u>The Incorporation of the schema encoding common modal axioms into axiomatic translation:</u> [1]

The schema encoding of common modal axioms is shown in table 2.2. The derivation of these translations is based upon normalization of the modal representation of the axiom by *limited* application of the relationships in formulae 2.4, in just the same way as has already been described in section 2.0. Again, π ('holds' in [1]) is a predicate (defined as shown below) under which propositional logical operators and the modal quantifier $\Box$ can be manipulated; p is any modal formula; x and y are distinct free variables; and R is the accessibility relation. This is simply another (minimal) version of the definition already seen in formulae 2.3.

$\forall p \forall x(\pi(\neg p,x) \leftrightarrow \neg\pi(p, x))$

$\forall pq \forall x(\pi(p*q,x) \leftrightarrow \pi(p, x)*\pi(q, x))$ where $* \in \{\rightarrow,\leftrightarrow,\vee,\wedge\}$

$\forall p \forall x(\pi(\Box p,x) \leftrightarrow \forall y(R(x,y) \rightarrow \pi(p, y)))$ [formulae 2.4, see section 2 in [1]]

The derivations for the axiomatic translation are given in figure 2.3. As already seen, the manipulations terminate still leaving all but the outermost modal operators in place (that is, *all* the inner modal operators in place). On the right hand side, there are still expressions in π, that need to be addressed, but not by inductive (recursive) processing. Instead, if the arguments of these expressions in π are then considered to be *ground terms*, a predicate may be substituted, so that for example $\pi(q,x)$ is written $Q_q(x)$ and $\pi(\Box p,y)$ is written $Q_{\Box p}(y)$. It is worth noting that the schemas encoding the axioms are not deterministic. Hence, for example, with axiom D (figure 2.3), alternative forms of the Q-predicate symbols can be formulated. The schema chosen is the form in which the minimum number of new Q-predicates needs to be introduced. This has consequences for any algorithm for automatically generating these schemas, since it will not be easy to ensure that these optimized forms are produced.

During translation, for each modal axiom (which is valid for a particular problem) a new formula is added ($\wedge$) to output. The origin of this formula is the translation of an *edited* version the instantiation set, in which only sub-formulae immediately below a box ($\Box$) symbol are included. Each sub-formula in this edited-instantiation set is translated by applying the formula in table 2.2. The edited-instantiation set of φ is described by

$\Box\psi \in Sf(\varphi)$, and is denoted by $\chi^{\in}_{\varphi}$

where $\chi^{\in}_{\varphi} = \{ \psi \mid \Box\psi \in Sf(\varphi) \}$; the superscript $^{\in}$ refers to the base state, with no modal axioms applied. The formal definition of this process is given in clause 2 of the definition of axiomatic translation in [1]. The schema for the particular modal axiom is *instantiated* for each member of the instantiation set.

The process is best illustrated by an example. Considering $\varphi = \Box((\neg\Box(r,p)) \wedge \Box(r,q))$ in KB, where the axiomatic translation of B is $\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box r,p}(y)) \vee Q_{r,p}(x))$. (This is a uni-modal case and so modality index r can be safely dropped, but R is still the predicate symbol associated with the modality index).

The edited instantiation set is

1. $\Box((\neg\Box p) \wedge \Box q)$ 2. $\Box p$ 3. $\Box q$

So φ in KB is

	$\prod(\Box((\neg\Box p) \wedge \Box q))$	[as defined above in section 3]
$\wedge$	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box(\neg\Box p \wedge \Box q)}(y)) \vee Q_{\neg\Box p \wedge \Box q}(x))$	$B(\Box(\neg\Box p \wedge \Box q))$
$\wedge$	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y)) \vee Q_p(x))$	$B(\Box p)$
$\wedge$	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box q}(y)) \vee Q_q(x))$	$B(\Box q)$

The notation is as follows: $B(\Box q)$ is an instance of the schema for axiom B instantiated for $\Box q$.

Table 2.2 also shows that, for the translation of some modal axioms, new symbols may need to be *Defined* – that is translated in $Def(\psi)$ [formula 2.2, see clause 3 of definition 4.1 of axiomatic translation in [1]], and the results again added ($\wedge$) to the output formula. The origin of these formulae is clear. For axiom D, the translation is $\neg Q_{\Box\phi}(x) \vee Q_{\neg\Box\neg\phi}(x)$. Here the predicate $Q_{\Box\phi}$ has been seen (and *Defined*) before, but the predicate $Q_{\neg\Box\neg\phi}$ is new. This arises from translation of the formula $\neg\Box\neg\phi$ in axiom D ($\Box\phi \rightarrow \neg\Box\neg\phi$). It has already been shown (section 2.0) that the result of Definition of negated and non-negated formulae is identical, and hence it is the new formula $\Box\neg\phi$ that needs to be Defined. Making this clearer by considering a further example, if the sub-formula under consideration is $\Box\Box q$, then the new formula Defined is $Def(\Box\neg(\Box q))$ or $Def(\Box\neg\Box q)$. In a similar way, for axiom

alt_1 ($\neg\Box p \rightarrow \Box\neg p$; $\neg Q_{\neg\Box\phi}(x) \vee Q_{\Box\neg\phi}(x)$), a new formula $Def(\Box\neg\phi)$ needs to be defined. The same argument applies for axiom $Alt_1^{\kappa 1\kappa 2}$ ($\neg\Box^{k1}\Box p \rightarrow \Box^{\kappa 2}\Box\neg p$; $\neg R^{\kappa 1}(x,y) \vee \neg Q_{\neg\Box p}(y) \vee \neg R^{\kappa 2}(x,z) \vee Q_{\Box\neg p}(z)$). No new formulae need to be defined for the other common axioms illustrated in table 2.2.

So continuing the running example, φ in KD is

	$\prod(\Box((\neg\Box p) \wedge \Box q))$	[as defined above in section 2.0]
$\wedge$	$\forall x(\neg Q_{\Box(\neg\Box p \wedge \Box q)}(x) \vee Q_{\neg\Box\neg(\neg\Box p \wedge \Box q)}(x))$	$[D(\Box(\neg\Box p \wedge \Box q))]$
$\wedge$	$\forall x(\neg Q_{\Box p}(x) \vee Q_{\neg\Box\neg p}(x))$	$[D(\Box p)]$
$\wedge$	$\forall x(\neg Q_{\Box q}(x) \vee Q_{\neg\Box\neg q}(x))$	$[D(\Box q)]$
$\wedge$	$\forall x(Q_{\Box\neg(\neg\Box p \wedge \Box q)}(x) \rightarrow \forall y(R(x, y) \rightarrow Q_{\neg(\neg\Box p \wedge \Box q)}(y)))$	$[Def(\Box\neg(\neg\Box p \wedge \Box q))]$
$\wedge$	$\forall x(Q_{\Box\neg(\neg\Box p \wedge \Box q)}(x) \leftrightarrow \neg Q_{\neg\Box\neg(\neg\Box p \wedge \Box q)}(x))$	
$\wedge$	$\forall x(Q_{\neg\Box\neg(\neg\Box p \wedge \Box q)}(x) \rightarrow \exists y(R(x, y) \wedge Q_{\neg\Box p \wedge \Box q}(y)))$	
$\wedge$	$\forall x(Q_{\Box\neg p}(x) \rightarrow \forall y(R(x, y) \rightarrow Q_{\neg p}(y)))$	$[Def(\Box\neg p)]$
$\wedge$	$\forall x(Q_{\Box\neg p}(x) \leftrightarrow \neg Q_{\neg\Box\neg p}(x))$	
$\wedge$	$\forall x(Q_{\neg\Box\neg p}(x) \rightarrow \exists y(R(x, y) \wedge Q_p(y)))$	
$\wedge$	$\forall x(Q_{\Box\neg q}(x) \rightarrow \forall y(R(x, y) \rightarrow Q_{\neg q}(y)))$	$[Def(\Box\neg q)]$
$\wedge$	$\forall x(Q_{\Box\neg q}(x) \leftrightarrow \neg Q_{\neg\Box\neg q}(x))$	
$\wedge$	$\forall x(Q_{\neg\Box\neg q}(x) \rightarrow \exists y(R(x, y) \wedge Q_q(y)))$	

The formal definition of these transformations is given in clause 2 of the definition 4.1 of axiomatic translation in [1].

Table 2.2 Schema encoding of some modal axioms for Axiomatic Translation (see figure 3 in [1], and section 1)

Axiom	Translation of $\Box(r,p)$	New symbols
T reflexive	$\forall x(\neg Q_{\Box r,p}(x) \vee Q_{r,p}(x))$	None
D serial	$\forall x(\neg Q_{\Box r,p}(x) \vee Q_{\neg\Box r,\neg p}(x))$	$\Box r,\neg p$
4 transitive	$\forall x(\neg Q_{\Box r,p}(x) \vee \forall y(\neg R(x,y) \vee Q_{\Box r,p}(y)))$ *	None
5 euclidean	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box r,p}(y)) \vee Q_{\Box r,p}(x))$ NB. see also section 2.1.2	None
B symmetry	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box r,p}(y)) \vee Q_{r,p}(x))$	None
alt_1 functional	$\forall x(\neg Q_{\neg\Box r,p}(x) \vee Q_{\Box r,\neg p}(x))$	$\Box r,\neg p$
4^{κ} 4^2 4^3	$\forall x(\neg Q_{\Box r,p}(x) \vee \forall y(\neg R^{\kappa}(x,y) \vee Q_{\Box r,p}(y)))$ * $\forall x(\neg Q_{\Box r,p}(x) \vee \forall yz(\neg R(x,y) \vee \neg R(y,z) \vee Q_{\Box r,p}(z)))$ $\forall x(\neg Q_{\Box r,p}(x) \vee \forall yzu(\neg R(x,y) \vee \neg R(y,z) \vee \neg R(z,u) \vee Q_{\Box r,p}(u)))$	None
5^{κ} 5^2 5^3	$\forall x(Q_{\Box r,p}(x) \vee \forall y(\neg R^{\kappa}(x,y) \vee \neg Q_{\Box r,p}(y)))$ NB. see also section 2.1.2 * $\forall x(Q_{\Box r,p}(x) \vee \forall zy(\neg R(x,z) \vee \neg R(z,y) \vee \neg Q_{\Box r,p}(y)))$ $\forall x(Q_{\Box r,p}(x) \vee \forall zuy(\neg R(x,z) \vee \neg R(z,u) \vee \neg R(u,y) \vee \neg Q_{\Box r,p}(y)))$	None
$alt_1^{\kappa 1\kappa 2}$ $alt_1^{1,1}$ $alt_1^{1,2}$ $alt_1^{2,1}$ $alt_1^{2,2}$	$\forall xyz(\neg R^{\kappa 1}(x,y) \vee \neg R^{\kappa 2}(x,z) \vee \neg Q_{\neg\Box r,p}(y) \vee Q_{\Box r,\neg p}(z))$ * $\forall x(\forall y[\neg R(x,y) \vee \neg Q_{\neg\Box r,p}(y)] \vee \forall z[\neg R(x,z) \vee Q_{\Box r,\neg p}(z)])$ $\forall x(\forall y[\neg R(x,y) \vee \neg Q_{\neg\Box r,p}(y)] \vee \forall uz[\neg R(x,u) \vee \neg R(u,z) \vee Q_{\Box r,\neg p}(z)])$ $\forall x(\forall uy[\neg R(x,u) \vee \neg R(u,y) \vee Q_{\neg\Box r,p}(y)] \vee \forall z[\neg R(x,z) \vee Q_{\Box r,\neg p}(z)])$ $\forall x(\forall uy[\neg R(x,u) \vee \neg R(u,y) \vee \neg Q_{\neg\Box r,p}(y)] \vee \forall vz[\neg R(x,v) \vee \neg R(v,z) \vee Q_{\Box r,\neg p}(z)])$	$\Box r,\neg p$

In this table, the modality index r has been included. In the uni-modal case, this index is normally excluded (see table figure 3 in [1]); There are other forms for some of expressions marked (*); Quantifiers can be extracted from the matrix, for example

$\forall xy(\neg R(x,y) \vee \neg Q_{\Box p}(y) \vee Q_{\Box p}(x))$ for axiom 5, and

$\forall xuyvz(\neg R(x,u) \vee \neg R(u,y) \vee \neg Q_{\neg\Box p}(y) \vee \neg R(x,v) \vee \neg R(v,z) \vee Q_{\Box\neg p}(z))$ for $Alt_1^{2,2}$.

Figure 2.3: <u>Derivation of schema encoding of common axioms for axiomatic translation:</u>

$\forall p \forall x(\pi(\neg p,x) \leftrightarrow \neg\pi(p, x))$ [1]

$\forall pq \forall x(\pi(p*q,x) \leftrightarrow (\pi(p, x)*\pi(q, x)))$ [2] where $* \in \{\rightarrow,\leftrightarrow,\vee,\wedge\}$

$\forall p \forall x(\pi(\Box p,x) \leftrightarrow \forall y(R(x,y) \rightarrow \pi(p, y)))$ [3] [again, formulae 2.4]

<u>Axiom T</u>: $\Box p \rightarrow p$

$\forall p \forall x(\pi(\Box p \rightarrow p,x) \equiv \forall p \forall x(\pi(\Box p,x) \rightarrow \pi(p,x))$ [from 2]

Re-writing (so $\pi(q,x)$ is $Q_q(x)$ and $\forall q$ is $\forall Q$, which can be taken as implicit) gives

$\forall x(Q_{\Box p}(x) \rightarrow Q_p(x)) \equiv$ **$\forall x(\neg Q_{\Box p}(x) \vee Q_p(x))$**

<u>Axiom 4</u>: $\Box p \rightarrow \Box\Box p$

$\forall p\forall x(\pi(\Box p \to \Box\Box p,x)$ $\equiv \forall p\forall x(\pi(\Box p,x) \to \pi(\Box\Box p,x))$ [from 2]
$\equiv \forall p\forall x(\pi(\Box p,x) \to \forall y(R(x,y) \to \pi(\Box p,y)))$ [from 3]
Re-writing gives $\forall x(Q_{\Box p}(x) \to \forall y(R(x,y) \to Q_{\Box p}(y))$
$\equiv \forall x(\neg Q_{\Box p}(x) \vee \forall y(\neg R(x,y) \vee Q_{\Box p}(y))$

<u>Axiom B</u>: $\neg\Box\neg\Box p \to p$
$\forall p\forall x(\pi(\neg\Box\neg\Box p \to p,x)$ $\equiv \forall p\forall x(\pi(\neg\Box\neg\Box p,x) \to \pi(p,x))$ [from 2]
$\equiv \forall p\forall x(\neg\pi(\Box\neg\Box p,x) \to \pi(p,x))$ [from 1]
$\equiv \forall p\forall x(\neg\forall y(R(x,y) \to \pi(\neg\Box p,y)) \to \pi(p,x))$ [from 3]
$\equiv \forall p\forall x(\neg\forall y(R(x,y) \to \neg\pi(\Box p,y)) \to \pi(p,x))$ [from 1]
Re-writing gives $\forall x(\neg(\forall y(R(x,y) \to \neg Q_{\Box p}(y))) \to Q_p(x))$
$\equiv \forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y)) \vee Q_p(x))$

<u>Axiom D</u>: $\Box p \to \neg\Box\neg p$
$\forall p\forall x(\pi(\Box p \to \neg\Box\neg p,x)$ $\equiv \forall p\forall x(\pi(\Box p,x) \to \pi(\neg\Box\neg p,x))$ [from 2]
$\equiv \forall p\forall x(\pi(\Box p,x) \to \neg\pi(\Box\neg p,x))$ [from 1]
Re-writing gives $\forall x(Q_{\Box p}(x) \to Q_{\neg\Box\neg p}(x))$
$\equiv \forall x(\neg Q_{\Box p}(x) \vee Q_{\neg\Box\neg p}(x))$
or $\forall x(Q_{\Box p}(x) \to \neg Q_{\Box\neg p}(x))$
$\equiv \forall x(\neg Q_{\Box p}(x) \vee \neg Q_{\Box\neg p}(x))$

<u>Axiom alt$_1$</u>: $\neg\Box p \to \Box\neg p$
$\forall p\forall x(\pi(\neg\Box p \to \Box\neg p,x)$ $\equiv \forall p\forall x(\pi(\neg\Box p,x) \to \pi(\Box\neg p,x))$ [from 2]
Re-writing gives $\forall x(Q_{\neg\Box p}(x) \to Q_{\Box\neg p}(x))$
$\equiv \forall x(\neg Q_{\neg\Box p}(x) \vee Q_{\Box\neg p}(x))$

<u>Axiom 5</u>: $\neg\Box\neg\Box p \to \Box p$
$\forall p\forall x(\pi(\neg\Box\neg\Box p \to \Box p,x)$ $\equiv \forall p\forall x(\pi(\neg\Box\neg\Box p,x) \to \pi(\Box p,x))$ [from 2]
$\equiv \forall p\forall x(\neg\pi(\Box\neg\Box p,x) \to \pi(\Box p,x))$ [from 1]
$\equiv \forall p\forall x(\neg\forall y(R(x,y) \to \pi(\neg\Box p,y)) \to \pi(\Box p,x))$ [from 3]
$\equiv \forall p\forall x(\neg\forall y(R(x,y) \to \neg\pi(\Box p,y)) \to \pi(\Box p,x))$ [from 1]
Re-writing gives $\forall x(\neg\forall y(R(x,y) \to \neg Q_{\Box p}(y)) \to Q_{\Box p}(x))$
$\equiv \forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y)) \vee Q_{\Box p}(x))$

<u>Axiom 4^K</u>: $\Box p \to \Box^K\Box p$
$\forall p\forall x(\pi(\Box p \to \Box^K\Box p,x)$ $\equiv \forall p\forall x(\pi(\Box p,x) \to \pi(\Box^K\Box p,x))$ [from 2]
$\kappa=1$ $\forall p\forall x(\pi(\Box p,x) \to \pi(\Box\Box p,x))$
$\equiv \forall p\forall x(\pi(\Box p,x) \to \forall y(R(x,y) \to \pi(\Box p,y)))$ [from 3]
$\kappa=2$ $\forall p\forall x(\pi(\Box p,x) \to \pi(\Box\Box\Box p,x))$
$\equiv \forall p\forall x(\pi(\Box p,x) \to \forall y(R(x,y) \to \pi(\Box\Box p,y)))$ [from 3]
$\equiv \forall p\forall x(\pi(\Box p,x) \to \forall y(R(x,y) \to \forall z(R(y,z) \to \pi(\Box p,z))))$ [from 3]
$\kappa=3$ $\forall p\forall x(\pi(\Box p,x) \to \pi(\Box\Box\Box\Box p,x))$
$\equiv \forall p\forall x(\pi(\Box p,x) \to \forall y(R(x,y) \to \pi(\Box\Box\Box p,y)))$
$\equiv \forall p\forall x(\pi(\Box p,x) \to \forall y(R(x,y) \to \forall z(R(y,z) \to \pi(\Box\Box p,z))))$ [from 3]
$\equiv \forall p\forall x(\pi(\Box p,x) \to \forall y(R(x,y) \to \forall z(R(y,z) \to \forall u(R(z,u) \to \pi(\Box p,u)))))$ [from 3]
Re-writing gives
$\kappa=1$ $\forall x(Q_{\Box p}(x) \to \forall y(R(x,y) \to Q_{\Box p}(y)))$
$\equiv \forall x(\neg Q_{\Box p}(x) \vee \forall y(\neg R(x,y) \vee Q_{\Box p}(y)))$
$\kappa=2$ $\forall x(Q_{\Box p}(x) \to \forall y(R(x,y) \to \forall z(R(y,z) \to Q_{\Box p}(z))))$
$\equiv \forall x(\neg Q_{\Box p}(x) \vee \forall y(\neg R(x,y) \vee \forall z(\neg R(y,z) \vee Q_{\Box p}(z))))$
$\kappa=3$ $\forall x(Q_{\Box p}(x) \to \forall y(R(x,y) \to \forall z(R(y,z) \to \forall u(R(z,u) \to Q_{\Box p}(u)))))$
$\equiv \forall x(\neg Q_{\Box p}(x) \vee \forall y(\neg R(x,y) \vee \forall z(\neg R(y,z) \vee \forall u(\neg R(z,u) \vee Q_{\Box p}(u)))))$
κ $\forall x(\neg Q_{\Box p}(x) \vee \forall y(\neg R^K(x,y) \vee Q_{\Box p}(y)))$

<u>Axiom 5^K</u>: $\neg\Box^K\neg\Box p \to \Box p$
$\forall p\forall x(\pi(\neg\Box^K\neg\Box p \to \Box p,x) \equiv \forall p\forall x(\pi(\neg\Box^K\neg\Box p,x) \to \pi(\Box p,x))$ [from 2]
$\kappa=1$ $\forall p\forall x(\pi(\neg\Box\neg\Box p,x) \to \pi(\Box p,x))$
$\equiv \forall p\forall x(\neg\pi(\Box\neg\Box p,x) \to \pi(\Box p,x))$ [from 1]
$\equiv \forall p\forall x(\neg\forall y(R(x,y) \to \pi(\neg\Box p,y)) \to \pi(\Box p,x))$ [from 3]
$\equiv \forall p\forall x(\neg\forall y(R(x,y) \to \neg\pi(\Box p,y)) \to \pi(\Box p,x))$ [from 1]
$\kappa=2$ $\forall p\forall x(\pi(\neg\Box\Box\neg\Box p,x) \to \pi(\Box p,x))$
$\equiv \forall p\forall x(\neg\pi(\Box\Box\neg\Box p,x) \to \pi(\Box p,x))$ [from 1]
$\equiv \forall p\forall x(\neg\forall y(R(x,y) \to \pi(\Box\neg\Box p,y)) \to \pi(\Box p,x))$ [from 3]
$\equiv \forall p\forall x(\neg\forall y(R(x,y) \to \forall u(R(y,u) \to \pi(\neg\Box p,u))) \to \pi(\Box p,x))$ [from 3]
$\equiv \forall p\forall x(\neg\forall y(R(x,y) \to \forall u(R(y,u) \to \neg\pi(\Box p,u))) \to \pi(\Box p,x))$ [from 1]
$\kappa=3$ $\forall p\forall x(\pi(\neg\Box\Box\Box\neg\Box p,x) \to \pi(\Box p,x))$
$\equiv \forall p\forall x(\neg\pi(\Box\Box\Box\neg\Box p,x) \to \pi(\Box p,x))$ [from 1]
$\equiv \forall p\forall x(\neg\forall y(R(x,y) \to \pi(\Box\Box\neg\Box p,y)) \to \pi(\Box p,x))$ [from 3]
$\equiv \forall p\forall x(\neg\forall y(R(x,y) \to \forall u(R(y,u) \to \pi(\Box\neg\Box p,u))) \to \pi(\Box p,x))$ [from 3]

$\equiv \forall p\forall x(\neg\forall y(R(x,y) \to \forall u(R(y,u) \to \forall v(R(u,v) \to \pi(\neg\Box p,v)))) \to \pi(\Box p,x))$ [from 3]
$\equiv \forall p\forall x(\neg\forall y(R(x,y) \to \forall u(R(y,u) \to \forall v(R(u,v) \to \neg\pi(\Box p,v)))) \to \pi(\Box p,x))$ [from 1]
Re-writing gives
$\kappa=1$ $\forall x(\neg\forall y(R(x,y) \to \neg Q_{\Box p}(y)) \to Q_{\Box p}(x))$
$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y)) \vee Q_{\Box p}(x))$
$\kappa=2$ $\forall x(\neg\forall y(R(x,y) \to \forall u(R(y,u) \to \neg Q_{\Box p}(u))) \to Q_{\Box p}(x))$
$\forall x(\forall y(\neg R(x,y) \vee \forall u(\neg R(y,u) \vee \neg Q_{\Box p}(u))) \vee Q_{\Box p}(x))$
$\kappa=3$ $\forall x(\neg\forall y(R(x,y) \to \forall u(R(y,u) \to \forall v(R(u,v) \to \neg Q_{\Box p}(v)))) \to Q_{\Box p}(x))$
$\forall x(\forall y(\neg R(x,y) \vee \forall u(\neg R(y,u) \vee \forall v(\neg R(u,v) \vee \neg Q_{\Box p}(v)))) \vee Q_{\Box p}(x))$
κ $\forall x(\forall y(\neg R^K(x,y) \vee \neg Q_{\Box p}(y)) \vee Q_{\Box p}(x))$

<u>Axiom alt$_1^{K1,K2}$</u>: $\neg\Box^{K1}\Box p \to \Box^{K2}\Box\neg p$
$\forall p\forall x(\pi(\neg\Box^{K1}\Box p \to \Box^{K2}\Box\neg p,x) \equiv \forall p\forall x(\pi(\neg\Box^{K1}\Box p,x) \to \pi(\Box^{K2}\Box\neg p,x))$ [from 2]
$\kappa1=1, \kappa2=1$ $\forall p\forall x(\pi(\neg\Box\Box p,x) \to \pi(\Box\Box\neg p,x))$
$\equiv \forall p\forall x(\neg\pi(\Box\Box p,x) \to \pi(\Box\Box\neg p,x))$ [from 1]
$\equiv \forall p\forall x(\neg\forall y(R(x,y) \to \pi(\Box p,y)) \to \forall z(R(x,z) \to \pi(\Box\neg p,z)))$ [from 3]
$\kappa1=1, \kappa2=2$ $\forall p\forall x(\pi(\neg\Box\Box p,x) \to \pi(\Box\Box\Box\neg p,x))$
$\equiv \forall p\forall x(\neg\forall y(R(x,y) \to \pi(\Box p,y)) \to \forall u(R(x,u) \to \pi(\Box\Box\neg p,u)))$ [from 1,3]
$\equiv \forall p\forall x(\neg\forall y(R(x,y) \to \pi(\Box p,y)) \to \forall u(R(x,u) \to \forall z(R(u,z) \to \pi(\Box\neg p,z))))$ [from 3]
$\kappa1=2, \kappa2=1$ $\forall p\forall x(\pi(\neg\Box\Box\Box p,x) \to \pi(\Box\Box\neg p,x))$
$\equiv \forall p\forall x(\neg\forall u(R(x,u) \to \pi(\Box\Box p,u)) \to \forall z(R(x,z) \to \pi(\Box\neg p,z)))$ [from 1,3]
$\equiv \forall p\forall x(\neg\forall u(R(x,u) \to \forall y(R(u,y) \to \pi(\Box p,y))) \to \forall z(R(x,z) \to \pi(\Box\neg p,z)))$ [from 3]
$\kappa1=2, \kappa2=2$ $\forall p\forall x(\pi(\neg\Box\Box\Box p,x) \to \pi(\Box\Box\Box\neg p,x))$
$\equiv \forall p\forall x(\neg\forall u(R(x,u) \to \pi(\Box\Box p,u)) \to \forall v(R(x,v) \to \pi(\Box\Box\neg p,v)))$ [from 1,3]
$\equiv \forall p\forall x(\neg\forall u(R(x,u) \to \forall y(R(u,y) \to \pi(\Box p,y))) \to \forall v(R(x,v) \to \forall z(R(v,z) \to \pi(\Box\neg p,z))))$ [from 3]
Re-writing gives
$\kappa1=1, \kappa2=1$ $\forall x(\neg\forall y(R(x,y) \to Q_{\Box p}(y)) \to \forall z(R(x,z) \to Q_{\Box\neg p}(z)))$
$\equiv \forall x(\forall y(\neg R(x,y) \vee Q_{\Box p}(y)) \vee \forall z(\neg R(x,z) \vee Q_{\Box\neg p}(z)))$
$\kappa1=1, \kappa2=2$ $\forall x(\neg\forall y(R(x,y) \to Q_{\Box p}(y)) \to \forall u(R(x,u) \to \forall z(R(u,z) \to Q_{\Box\neg p}(z))))$
$\equiv \forall x(\forall y(\neg R(x,y) \vee Q_{\Box p}(y)) \vee \forall u(\neg R(x,u) \vee \forall z(\neg R(u,z) \vee Q_{\Box\neg p}(z))))$
$\kappa1=2, \kappa2=1$ $\forall x(\neg\forall u(R(x,u) \to \forall y(R(u,y) \to Q\Box p(y))) \to \forall z(R(x,z) \to Q\Box\neg p(z)))$
$\equiv \forall x(\forall u(\neg R(x,u) \vee \forall y(\neg R(u,y) \vee Q\Box p(y))) \vee \forall z(\neg R(x,z) \vee Q\Box\neg p(z)))$
$\kappa1=2, \kappa2=2$ $\forall x(\neg\forall u(R(x,u) \to \forall y(R(u,y) \to Q_{\Box p}(y))) \to \forall v(R(x,v) \to \forall z(R(v,z) \to Q_{\Box\neg p}(z))))$
$\equiv \forall x(\forall u(\neg R(x,u) \vee \forall y(\neg R(u,y) \vee Q_{\Box p}(y))) \vee \forall v(\neg R(x,v) \vee \forall z(\neg R(v,z) \vee Q_{\Box\neg p}(z))))$
$\kappa1,\kappa2$ $\forall x(\forall y(\neg R^{K1}(x,y) \vee Q_{\Box p}(y)) \vee \forall z(\neg R^{K2}(x,z) \vee Q_{\Box\neg p}(z)))$

It is worth noting that while the treatment here seems to make an implicit link between correspondence properties and the incorporation of modal schema in the axiomatic translation, this is for narrative purposes only. Many modal axioms do not have correspondence properties, but can be successfully subject in turn to axiomatic translation, and first order resolution. A case in point is axiom M (McKinsey, $\Box\neg\Box\neg p \to \neg\Box\neg\Box p$). The schema encoding of this axiom is considered in the results section.

2.1.3. <u>Composition in the incorporation of several modal axioms in axiomatic translation</u> (see section 5 in [1]).

As seen in section 1, it is commonly necessary for several modal axioms to be valid in a restricted set of Kripke frames in order to model a useful property. How are these combinations of axioms handled in axiomatic translation? In many cases, formulae arising from the translation of a set of sub-formulae are just added ($\wedge$) together in the final output formula with no interaction between the modal axioms in the translation process. This is the case for combinations of axioms listed below. For proofs, see [1]. Importantly, note that since there is no interaction between the modal axioms during the translation, the *order* in which the modal axioms are applied to a set of sub-formulae is *unimportant*.

- KT4, KTB, KDB, KD4.

So pursuing the running example, φ in KTB is
$\prod(\Box(\neg\Box p \wedge \Box q))$ [as defined above in section 2.0]
$\wedge$ $\forall x(\neg Q_{\Box(\neg\Box p\wedge\Box q)}(x) \vee Q_{\neg\Box p\wedge\Box q}(x))$ $T(\Box(\neg\Box p\wedge\Box q))$
$\wedge$ $\forall x(\neg Q_{\Box p}(x) \vee Q_p(x))$ $T(\Box p)$
$\wedge$ $\forall x(\neg Q_{\Box q}(x) \vee Q_q(x))$ $T(\Box q)$
$\wedge$ $\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box(\neg\Box p\wedge\Box q)}(y)) \vee Q_{\neg\Box p\wedge\Box q}(x))$ $B(\Box(\neg\Box p\wedge\Box q))$
$\wedge$ $\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y)) \vee Q_p(x))$ $B(\Box p)$
$\wedge$ $\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box q}(y)) \vee Q_p(x))$ $B(\Box q)$

and more interestingly, φ in KDB is
$\prod(\Box(\neg\Box p \wedge \Box q))$ [as defined above in section 2.0]
$\wedge$ $\forall x(\neg Q_{\Box(\neg\Box p\wedge\Box q)}(x) \vee Q_{\neg\Box\neg(\neg\Box p\wedge\Box q)}(x))$ $D(\Box(\neg\Box p\wedge\Box q))$

∧ $\forall x(\neg Q_{\Box p}(x) \vee Q_{\neg\Box\neg p}(x))$ D(□p)

∧ $\forall x(\neg Q_{\Box q}(x) \vee Q_{\neg\Box\neg q}(x))$ D(□q)

∧ Def(□¬(¬□p∧□q)) ∧ Def(□¬p) ∧ Def(□¬q) *

∧ $\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box(\neg\Box p\wedge\Box q)}(y)) \vee Q_{\neg\Box p\wedge\Box q}(x))$ B(□(¬□p∧□q))

∧ $\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y)) \vee Q_{p}(x))$ B(□p)

∧ $\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box q}(y)) \vee Q_{p}(x))$ B(□q)

* Note in that while the new formulae arising from axiom B need to be Defined (as described above), they are *not* added to the instantiation set considered when axiom B is applied.

Unfortunately, this simple procedure does not represent the general case. In the list above, the combinations of axioms K4B and KT4B are not included since these combinations require *composition* of axiom 4 in these combinations (see below). In addition, the cases K5 and $K5^{\kappa}$ have complications that are related to this problem of composition.

First, consider the case K5. For axiom 5, the base-case of the instantiation set ($\chi^{\in}_{\varphi}$) of sub-formula immediately below a box (□) symbol is *not sufficient* to ensure a complete in the translation. An additional symbol of form □¬□ψ for each sub-formula □ψ must (i) be included in the instantiation set, and (ii) Defined, so:

$$\Box\chi^{5}_{\varphi} = \Box\chi^{\in}_{\varphi} \cup \Box\neg\Box\chi^{\in}_{\varphi}$$

Returning to the running example, φ in K5 is

∏(□(¬□p∧□q)) [as defined above in section 2.0]

∧ $\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box(\neg\Box p\wedge\Box q)}(y)) \vee Q_{\Box(\neg\Box p\wedge\Box q)}(x))$ ∧ Def(□¬□(¬□p∧□q)) [5(□(¬□p∧□q))]

∧ $\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box\neg\Box(\neg\Box p\wedge\Box q)}(y)) \vee Q_{\Box\neg\Box(\neg\Box p\wedge\Box q)}(x))$ [5(□¬□(¬□p∧□q)) from □(¬□p∧□q)]

∧ $\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y)) \vee Q_{\Box p}(x))$ ∧ Def(□¬□p) [5(□p)]

∧ $\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box\neg\Box p}(y)) \vee Q_{\Box\neg\Box p}(x))$ [5(□¬□p) from □p]

∧ $\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box q}(y)) \vee Q_{\Box q}(x))$ ∧ Def(□¬□q) [5(□q)]

∧ $\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box\neg\Box q}(y)) \vee Q_{\Box\neg\Box q}(x))$ [5(□¬□q) from □q]

(Note, it is sometimes convenient to refer to the default translation of axiom 5 as $K5_o$ in order to highlight the complication that exists here; if this terminology is being used then K5 represents the translation without these compositional formulae. This translation K5 may be used in experiments – but is not complete under resolution and so may not give the correct answer for the translation of any particular target modal formula).

In many cases, the satisfiablity of the formula φ is not affected if these additional formula are not included in the instantiation set (as in the above example, which is satisfiable for both K5 and $K5_o$), but a case from [1] that discriminates has be included in the test suite: ¬□□□p∧□□p* (in K5 satisfiable; but in $K5_o$ unsatisfiable).

$\Box\chi^{5}_{\varphi}$ is {□□□p, □□p, □p }, and Sf(φ) = {¬□□□p∧□□p, *□□□p, □□p, □p*, p }

and the translation is ∏(¬□□□p∧□□p)

∧ 5(□□□p) ∧ 5(□□p) ∧5(□p)

∧ Def(□¬□□□p) ∧ Def(□¬□□p) ∧ Def(□¬□p) ∧ 5(□¬□□□p) ∧ 5(□¬□□p) ∧ 5(□¬□p)

During this study other cases in which these formulae are critical were found: ◇◇◇◇◇◇¬q∧◇p∧□□p, ◇◇◇◇◇¬q∧□□□q, ◇◇◇◇¬q∧□□p, ◇◇◇◇¬q∧◇p∧□□p, ◇◇◇¬q∧□□q, ◇◇◇¬q∧◇p∧□□p (these are *counterexamples* for translations without the additional compositional formulae).

Translation of axiom 5 is further illustrated by the following example where, formulae contributing to translation of axiom 5 are highlighted, formulae making this translation complete are marked 5o, and the contribution of new compositional formulae to the instantiation set is illustrated by the translation of axiom T.

Input … ¬(□(r,p)→p) in $K5_o$T

```
Standardized … □(r,p)∧¬p                  or ∧(□(r,p),¬p)
               Instantiation set … { □(r,p), p, □(r,p)∧¬p }
-------------------------------------------------------------------------------
Def(□(r,p))                        ∀x(Q_□rp(x) ↔ ¬Q_¬□rp(x))                      ↑
                                 ∧ ∀x(Q_□rp(x) → ∀y(R_r(x,y) → Q_p(y)))           |
                                 ∧ ∀x(Q_¬□rp(x) → ∃y(R_r(x,y) ∧ Q_¬p(y)))         |
Def(∧(□(r,p),¬(p)))          ∧ ∀x(Q_∧□rp¬p(x) ↔ ¬Q_¬∧□rp¬p(x))                    |  K
                                 ∧ ∀x(Q_∧□rp¬p(x) → (Q_¬p(x) ∧ Q_□rp(x)))         |
                                 ∧ ∀x(Q_¬∧□rp¬p(x) → (Q_p(x) ∨ Q_¬□rp(x)))        |
Def(p)                       ∧ ∀x(Q_p(x) ↔ ¬Q_¬p(x))                              ↓
-------------------------------------------------------------------------------
               □ Instantiation set … { □(r,p) }                                   ↕  5
5(□(r,p))                        ∧ ∀x(Q_□rp(x) ∨ ∀y(¬R_r(x,y) ∨ ¬Q_□rp(y)))
-------------------------------------------------------------------------------
5(□(r,¬(□(r,p))))            ∧ ∀x(Q_□r¬□rp(x) ∨ ∀y(¬R_r(x,y) ∨ ¬Q_□r¬□rp(y)))    5o
Def(□(r,¬(□(r,p))))          ∧ ∀x(Q_□r¬□rp(x) ↔ ¬Q_¬□r¬□rp(x))
                                 ∧ ∀x(Q_□r¬□rp(x) → ∀y(R_r(x,y) → Q_¬□rp(y)))
                                 ∧ ∀x(Q_¬□r¬□rp(x) → ∃y(R_r(x,y) ∧ Q_□rp(y))
-------------------------------------------------------------------------------
               □ Instantiation set … { □(r,p), □(r,¬(□(r,p)))}                    ↑
T(□(r,p))                        ∧ ∀x(¬Q_□rp(x) ∨ Q_p(x))                         |  T
T(□(r,¬(□(r,p))))            ∧ ∀x(¬Q_□r¬□rp(x) ∨ Q_¬□rp(x))                       ↓
```


Similar considerations apply to axiom 5^{κ} (see table 2.6). For ¬□□□p∧□□p in $K5^2$ the translation is:

∏(¬□□□p∧□□p)

∧ 5^2(□□□p) ∧ 5^2(□□p) ∧5^2(□p)

∧ Def(□□¬□□□p) ∧ Def(□□¬□□p) ∧ Def(□□¬□p)∧ Def(□¬□□□p)* ∧ Def(□¬□□p)* ∧ Def(□¬□p)*

∧ 5^2(□□¬□□□p) ∧ 5^2(□□¬□□p) ∧ 5^2(□□¬□p)∧ 5^2(□¬□□□p)* ∧ 5^2(□¬□□p)* ∧ 5^2(□¬□p)*

and for ¬□□□p∧□□p in $K5^3$ the translation is

∏(¬□□□p∧□□p)

∧ 5^3(□□□p) ∧ 5^3(□□p) ∧5^3(□p) ∧ Def(□□□¬□□□p) ∧ Def(□□□¬□□p) ∧ Def(□□□¬□p)

∧ Def(□□¬□□□p)* ∧ Def(□□¬□□p)* ∧ Def(□□¬□p)*∧ Def(□¬□□□p)* ∧ Def(□¬□□p)* ∧ Def(□¬□p)*

∧ 5^3(□□□¬□□□p) ∧ 5^3(□□□¬□□p) ∧ 5^3(□□□¬□p)

∧ 5^3(□□¬□□□p)* ∧ 5^3(□□¬□□p)* ∧ 5^3(□□¬□p)*∧ 5^3(□¬□□□p)* ∧ 5^3(□¬□□p)* ∧ 5^3(□¬□p)*

The formal definition of these additional formulae * in these translations is found in clause 3 of the definition of axiomatic translation in [1]. They essentially arise by considering all the sub-formulae in the new compositional formulae. In the case of axiom 5, no additional clauses arise from considering all the sub-formulae of the newly defined formulae (only □¬□ψ arises from □ψ). In the case of axiom 5^2 and axiom 5^3, a series of new formulae arise. For 5^2 □□¬□ψ, □¬□ψ arise from □ψ, and for 5^3 all of □□□¬□ψ, □□¬□ψ, □¬□ψ arise from □ψ.

It is possible to view the compositional terms for axioms 5 and 5^{κ} to be an 'internal' composition, isolated within the axiom schema itself. Composition can also take place for sequences of modal axioms – new compositional formulae in the translation arising in the interactions between axioms. Hence, in the same way as seen above, for the axiom combination K4B, the base-case of the instantiation set ($\chi^{\in}_{\varphi}$) is not sufficient. An additional formula of form □□ψ for each sub-formula □ψ must be included. The composition of a series of modal axioms can be described in formulae of χ^{α}_{φ} as follows [from 1] where φ is a modal formula

$$\chi^{\in}_{\varphi} = \{ \psi \mid \Box\psi \in Sf(\varphi) \}$$

$$\chi^{\alpha.A}_{\varphi} = \chi^{\alpha}_{\varphi} \cup \{ \phi\{p/\psi\} \mid \Box\phi \in \Box\chi^{\in}_{A}, \Box\psi \in Sf(\Box\chi^{\alpha}_{\varphi}) \}$$ [formulae 3.6, see section 5 in [1]).]

- □ψ is the set of box formulae in the sub formulae of φ, and ψ is the same set without the leading box.
- $\Box\chi^{\in}_{\varphi}$ is the base instantiation set with □ at the top of the formula, with no axioms applied
- where ∈ is the empty sequence of axioms (that is, just axiom K).
- $\Box\chi^{\alpha}_{\varphi}$ is the instantiation set, plus formulae induced by the axioms α
- where α is an *ordered* sequence of axioms
- $\Box\chi^{\alpha.A}_{\varphi}$ is the instantiation set, with the sequence of modal axioms α and then axiom A, in that order.
- and the substitution refers to the free variable p in formulae in table 2.2.

There is a difference from the composition in the case axiom 5^{κ}. Here multiple axioms are being applied in sequence, and the translation of an axiom may influence the instantiation set for the *next* axiom. That is, for axioms 5 and 5^{K}, the update of the instantiation set occurs *before* translation of the axiom, and in other cases the update of the instantiation set only effects the application of *subsequen*t axioms (*not* the current axiom).

For the case of axiom combination K4B, the instantiation sets for the translation are $\chi_4=\chi^{\in}_{\varphi}$ and $\chi_B=\chi^{4}_{\varphi}$ [table 2.5]. An example that has been included in the test suite is φ=¬(□¬□p∨□p) in K4B [mentioned in 1]. Here Sf(φ) = { ¬□¬□p∧¬□p, □¬□p, □p, p }, □χ = { □¬□p, □p }

and the translation is = ∏(□¬□p∨□p)

∧ 4(□¬□p) ∧ 4(□p) ∧ Def(□□¬□p) ∧ Def(□□p) **

∧ B(□¬□p) ∧ B(□p) ∧ B(□□¬□p) ∧ B(□□p)

Considering the sequence of transformations that take place, at point **, the instantiation set is updated to include the new formulae as follows: □χ = { □¬□p, □p, □□¬□p, □□p }. How is this derived?

$\varphi = \Box\neg\Box p \vee \Box p$, and $Sf(\varphi) = \{\Box\neg\Box p \vee \Box p, \Box\neg\Box p, \Box p, p\}$.

In formulae 2.5, as before, $\chi^{\in}_{\varphi} = \{ \neg\Box p, p \}$, and $\Box\chi^{\in}_{\varphi} = \{ \Box\neg\Box p, \Box p \}$. and $\psi = \{ \neg\Box p, p \}$, and $\Box\psi = \{ \Box\neg\Box p, \Box p \}$. This is the instantiation set when axiom 4 is translated (that is, $\chi_4 = \chi^{\in}_{\varphi}$). However, the instantiation set must be updated at the point ** *in light of* axiom 4.

$$\text{Then } \chi^{4}_{\varphi} = \chi^{\in.4}_{\varphi} = \chi^{\in}_{\varphi} \cup \{ \phi\{p/\psi\} \mid \Box\phi \in \Box\chi^{\in}_{4}, \Box\psi \in Sf(\Box\chi^{\in}_{\varphi}) \}.$$

It is this instantiation set that is used during the translation of axiom B (that is, $\chi_B = \chi^{4}_{\varphi}$). To construct this instantiation set, then it is necessary to recall that axiom 4 is: $\neg\Box p \vee \Box\Box p$. Substituting ψ in $\Box\chi^{\varepsilon}_{4}$, i.e. $\Box\psi \rightarrow \Box\Box\psi$ gives:

$\neg\Box(\neg\Box p)$	$\vee$	$\Box\Box(\neg\Box p)$	$\{\Box(\neg\Box p), \Box\Box(\neg\Box p)\}$
$\neg\Box(p)$	$\vee$	$\Box\Box(p)$	$\{\Box(p), \Box\Box(p)\}$

so $\Box\phi \in \{ \Box\neg\Box p, \Box\Box\neg\Box p, \Box p, \Box\Box p \}$ and $\phi \in \{ \neg\Box p, \Box\neg\Box p, p, \Box p \}$

and $\chi^{4.}_{\varphi} = \{ \neg\Box p, p \} \cup \{ \neg\Box p, \Box\neg\Box p, p, \Box p \}$
$= \{ \neg\Box p, \Box\neg\Box p, p, \Box p \}$

Finally, $\Box\chi^{4}_{\varphi} = \{ \Box\neg\Box p, \Box\Box\neg\Box p, \Box p, \Box\Box p \}$, in which the new formulae are $\{ \Box(\Box\neg\Box p), \Box(\Box p) \}$ arising from $\Box\psi = \{\Box\neg\Box p, \Box p \}$ respectively.

In the general case new formulae arising are of form $\Box\Box\psi$ for each sub-formula $\Box\psi$.

It can be seen that these new formulae need to be (i) Defined, and (ii) also added to the instantiation set for the translation of axiom B (that is, $\chi_B = \chi^{4}_{\varphi}$).

Note that it is sometimes convenient to refer to the default translation of axiom sequence K4B as $K4_oB$ to denote compostion It should also be apparent that there is considerable scope for expressions to be re-Defined during the manipulations described above, and a simple extension to the translation procedure, is to define a *set* of formulae, so that there is no duplication in effort, or additional formulae added needlessly to the first order formula. If such duplication were allowed, it would be likely to slow down or extend the proof by resolution.

Other counter-examples that are worth examining for the combination of axioms $K4_oB$ were identified in this study: $\Diamond\Box\Diamond\Box\Diamond\Box\Diamond\Box p \wedge q$, $\Diamond\Box\Diamond\Box\Diamond\Box\Diamond\Box\Diamond\Box p \wedge q$, $\neg(\neg\Box\neg\Box p \rightarrow \Box p)$, $\neg(\Diamond\Box(\Box p \rightarrow q) \rightarrow \Box(\Box p \rightarrow \Diamond q))$, $\neg(\neg\Box(\Box p \rightarrow q) \rightarrow \Box(\Box p \rightarrow \neg\Box(\Box p \rightarrow q)))$, $\Diamond\neg q \wedge \Diamond\Box q$. Similar considerations apply to the combination $KT4_oB$. Interesting counter-examples are again $\neg(\neg\Box\neg\Box p \rightarrow \Box p)$, $\neg(\Diamond\Box(\Box p \rightarrow q) \rightarrow \Box(\Box p \rightarrow \Diamond q))$, $\neg(\neg\Box(\Box p \rightarrow q) \rightarrow \Box(\Box p \rightarrow \neg\Box(\Box p \rightarrow q)))$, $\Diamond\neg q \wedge \Diamond\Box q$.

Other combinations of axioms, and the default instantiation sets required to ensure completeness have been described in [1]. They are listed in table 2.5. In [1], proofs are offered for the instantiation sets listed in table 2.5, and in some cases the positive shortcuts (section 2.1.4) that are usually considered optional must be included in the translation in order to ensure completeness.

It appears that the combination of axioms has a more general result (can be inferred from [1], but not proven). Simply that it is possible to define a compositional mechanism specifically tuned to each of the modal axioms, in which new sub-formulae are added to the instantiation set as each modal axiom is applied. Clearly, the *order* in which axioms are applied is important. The translation of an axiom must process the compositional sub-formulae arising from axioms applied *before* it, and will in turn contribute new sub-formulae that need to be processed by the axioms that are applied *after* it. Table 2.6 lists these new sub-formulae associated with each modal axiom in composition [some are given in [1], others are derived here]. Axioms 4_o and 5_o appear as already described. It should be clear that for the translation of axiom sequence $K\alpha_1 5_o \alpha_2$, the instantiation set is updated with new compositional formulae *before* translation of axiom 5, and the subsequent axioms (α_2) also need to take into account the extra formulae (in the form of $\neg\Box p$ added formulae to the instantiation set). A sketch of some of the derivations of table 2.6 is given in figure 2.7. Only a limited number are given, since it soon becomes apparent that deriving the general case is very mechanical, requiring only inspection of the modal form of the axiom and searching for formulae other than those of the form $\Box p$ and p. Hence axiom D is : $\Box p \rightarrow \neg\Box\neg p$ and eliminating $\Box p$, leaves $\Box\neg p$ for the compositional formulae (recall that the state of negation of these formulae is irrelevant).

It is worth noting that in principle, all axiom combinations could use composition, with additional formulae in the instantiation set arising from previously applied axioms. The result of resolution would be identical for combinations like KT4 and KTB (trivially equal because there are no compositional formulae for

axiom T), and KD_oB = KDB and KD_o4 = KD4. The inclusion of more formulae (that is, *unnecessary* formulae) in the translation passed to the SPASS resolution prover extends the problem to be considered, and may therefore lengthen the proof procedure.

Table 2.5 Axiomatic translation with combinations of axioms [gathered from [1]].

Theorem in [1]	Axiom combination	Instantiation set χ_A	Include positive shortcuts
Theorem 5.1	K4	$\chi_4 = \chi^{\in}_{\varphi}$	optional
	KT	$\chi_T = \chi^{\in}_{\varphi}$	
	KD	$\chi_D = \chi^{\in}_{\varphi}$	
	KB	$\chi_B = \chi^{\in}_{\varphi}$	
	$Kalt_1$	$\chi_{alt1} = \chi^{\in}_{\varphi}$	
Theorem 5.2	$K4^K$	$\chi_{4K} = \chi^{\in}_{\varphi}$	optional
	$Kalt1^{KK}$	$\chi_{alt1kk} = \chi^{\in}_{\varphi}$	
Theorem 5.3	KT4 (S4)	$\chi_T = \chi_4 = \chi^{\in}_{\varphi}$	optional
	KTB	$\chi_T = \chi_B = \chi^{\in}_{\varphi}$	
	KDB	$\chi_D = \chi_B = \chi^{\in}_{\varphi}$	
	KD4	$\chi_D = \chi_4 = \chi^{\in}_{\varphi}$	
Theorem 5.4	K5	$\chi_5 = \chi^{5}_{\varphi} = \chi^{\in}_{\varphi} \cup \neg\Box\chi^{\in}_{\varphi}$	required
Theorem 5.5	$K5^K$	$\chi_{5K} = \chi^{5K}_{\varphi} = \chi^{\in}_{\varphi} \cup \Box^{K-1}\neg\Box\chi^{\in}_{\varphi}$	required
Theorem 5.6	K4B	$\chi_4 = \chi^{\in}_{\varphi}$ and $\chi_B = \chi^{4}_{\varphi}$	required
Theorem 5.7	KT4B (S5)	$\chi_T = \chi_4 = \chi^{\in}_{\varphi}$ and $\chi_B = \chi^{4}_{\varphi}$	required

Table 2.6. Composition of the axiomatic translation of common modal axioms
developed from material in [1]

Axiom	Modal representation of axiom	For composition
T_o	$\Box p \rightarrow p$	Nothing
B_o	$\neg\Box\neg\Box p \rightarrow p$	$\Box\neg\Box p$
D_o	$\Box p \rightarrow \neg\Box\neg p$	$\Box\neg p$
4^K_o 4_o 4^2_o 4^3_o	$\Box p \rightarrow \Box k\Box p$ $\Box p \rightarrow \Box\Box p$ $\Box p \rightarrow \Box\Box\Box p$ $\Box p \rightarrow \Box\Box\Box\Box p$	$\Box^K\Box p$ $\Box\Box p$ $\Box\Box\Box p$ $\Box\Box\Box\Box p$
5^K_o 5_o 5^2_o 5^3_o	$\neg\Box k\neg\Box p \rightarrow \Box p$ $\neg\Box\neg\Box p \rightarrow \Box p$ $\neg\Box\Box\neg\Box p \rightarrow \Box p$ $\neg\Box\Box\Box\neg\Box p \rightarrow \Box p$	$\Box^K\neg\Box p$ $\Box\neg\Box p$ $\Box\Box\neg\Box p$ $\Box\Box\Box\neg\Box p$
$alt_1{}^{K1K2}{}_o$ alt_{1o} $alt_1{}^{1,1}{}_o$ $alt_1{}^{1,2}{}_o$ $alt_1{}^{2,1}{}_o$ $alt_1{}^{2,2}{}_o$	$\neg\Box^{K1}\Box p \rightarrow \Box^{K2}\Box\neg p$ $\neg\Box p \rightarrow \Box\neg p$ $\neg\Box\Box p \rightarrow \Box\Box\neg p$ $\neg\Box\Box p \rightarrow \Box\Box\Box\neg p$ $\neg\Box\Box\Box p \rightarrow \Box\Box\neg p$ $\neg\Box\Box\Box p \rightarrow \Box\Box\Box\neg p$	$\Box^{K2}\Box\neg p$ & $\Box^{K1}\Box p$ $\Box\neg p$ $\Box\Box p$ & $\Box\Box\neg p$ $\Box\Box p$ & $\Box\Box\Box\neg p$ $\Box\Box\Box p$ & $\Box\Box\neg p$ $\Box\Box\Box p$ & $\Box\Box\Box\neg p$

Figure 2.7. <u>Derivation of composition of the axiomatic translation of some common modal axioms</u>

The derivation of compositional formulae for axioms is illustrated with a simple example $\varphi = \Box p$. The general (table 2.6) case can be inferred from each result.

The following hold in every case.

$Sf(\varphi) = \{\Box p, p\}$, $\Box\chi = \{\Box p\}$, $\chi^{\in}_{\varphi} = \{p\}$, $\Box\chi^{\in}_{\varphi} = \{\Box p\}$, $\psi \in \{p\}$, $\Box\psi \in \{\Box p\}$

<u>$\varphi = \Box p$ in T_o:</u>

$\chi^{T}_{\varphi} = \chi^{\in.T}_{\varphi} = \chi^{\in}_{\varphi} \cup \{ \phi\{p/\psi\} \mid \Box\phi \in \Box\chi^{\in}_{T}, \Box\psi \in Sf(\Box\chi^{\in}_{\varphi}) \}$.

Substituting ψ in $\Box\chi^{\varepsilon}_{4}$, for $\neg\Box\psi \vee \psi$ for axiom T.

$\neg\Box p \vee p$ $\{\Box p\}$

so $\Box\phi \in \{\Box p\}$ and $\phi \in \{p\}$

and $\chi^{T}_{\varphi} = \{p\} \cup \{p\} = \{p\}$

Finally, $\Box\chi^{T}_{\varphi} = \{\Box p\}$ with no new formulae.

<u>$\varphi = \Box p$ in B_o:</u>

$\chi^{B}_{\varphi} = \chi^{\in.B}_{\varphi} = \chi^{\in}_{\varphi} \cup \{ \phi\{p/\psi\} \mid \Box\phi \in \Box\chi^{\in}_{B}, \Box\psi \in Sf(\Box\chi^{\in}_{\varphi}) \}$.

Substituting ψ in $\Box\chi^{\varepsilon}_{B}$, for $\Box\neg\Box\psi \vee \psi$ for axiom B.

$\Box\neg\Box p \vee p$ $\{\Box\neg\Box p\}$

so $\Box\phi \in \{\Box\neg\Box p\}$ and $\phi \in \{\neg\Box p\}$

<u>$\varphi = \Box p$ in D_o:</u>

$\chi^{D}_{\varphi} = \chi^{\in.D}_{\varphi} = \chi^{\in}_{\varphi} \cup \{ \phi\{p/\psi\} \mid \Box\phi \in \Box\chi^{\in}_{D}, \Box\psi \in Sf(\Box\chi^{\in}_{\varphi}) \}$.

Substituting ψ in $\Box\chi^{\varepsilon}_{D}$, for $\neg\Box\psi \vee \neg\Box\neg\psi$ for axiom D.

$\neg\Box p \vee \neg\Box\neg p$ $\{\Box\neg p, \Box p\}$

so $\Box\phi \in \{\Box\neg p, \Box p\}$ and $\phi \in \{\neg p, p\}$

and $\chi^{T}_{\varphi} = \{p\} \cup \{\neg p, p\} = \{\neg p, p\}$

Finally, $\Box\chi^{T}_{\varphi} = \{\Box\neg p, \Box p\}$ with new formulae of the form $\Box\neg p$.

<u>$\varphi = \Box p$ in 5_o:</u>

$\chi^{5}_{\varphi} = \chi^{\in.5}_{\varphi} = \chi^{\in}_{\varphi} \cup \{ \phi\{p/\psi\} \mid \Box\phi \in \Box\chi^{\in}_{5}, \Box\psi \in Sf(\Box\chi^{\in}_{\varphi}) \}$.

Substituting ψ in $\Box\chi^{\varepsilon}_{5}$, for $\Box\neg\Box\psi \vee \Box\psi$ for axiom 5_o.

$\Box\neg\Box p \vee \Box p$ $\{ \Box p, \Box\neg\Box p \}$

so $\Box\phi \in \{\Box p, \Box\neg\Box p\}$ and $\phi \in \{p, \neg\Box p\}$

and $\chi^{B}_{\varphi} = \{ p \} \cup \{ \neg\Box p \} = \{ \neg\Box p, p\}$

Finally, $\Box\chi^{B}_{\varphi} = \{\Box\neg\Box p, \Box p\}$ with new formulae of the form $\Box\neg\Box p$.

<u>$\varphi=\Box p$ in alt_{1o}:</u>

$\chi^{alt1}_{\varphi} = \chi^{\in.alt1}_{\varphi} = \chi^{\in}_{\varphi} \cup \{\phi \{p/\psi\} \mid \Box\phi \in \Box\chi^{\in}_{alt1}, \Box\psi \in Sf(\Box\chi^{\in}_{\varphi}) \}$.

Substituting ψ in $\Box\chi^{\varepsilon}_{alt1}$, for $\Box\psi \vee \Box\neg\psi$ for axiom alt_1.

$\Box p \vee \Box\neg p$ $\{\Box\neg p, \Box p\}$

so $\Box\phi \in \{\Box\neg p, \Box p\}$ and $\phi \in \{\neg p, p\}$

and $\chi^{alt1}_{\varphi} = \{p\} \cup \{\neg p, p\} = \{\neg p, p\}$

Finally, $\Box\chi^{alt1}_{\varphi} = \{\Box\neg p, \Box p\}$ with new formulae of the form $\Box\neg p$.

and $\chi^{5}_{\varphi} = \{p\} \cup \{ p, \neg\Box p \} = \{p, \neg\Box p\}$

Finally, $\Box\chi^{5}_{\varphi} = \{\Box p, \Box\neg\Box p\}$ with new formulae of the form $\Box\neg\Box p$.

<u>$\varphi=\Box p$ in 4_o:</u>

see the text

2.1.4. <u>Shortcuts.</u> [1]

Shortcuts have already been mentioned in formula 2.2, where the default definition contains the expression $\forall x(Q_{\psi}(x)\rightarrow\neg Q_{\sim\psi}(x))$, extended by conjugation with $\forall x(Q_{\psi}(x)\leftarrow\neg Q_{\sim\psi}(x))$, to $\forall x(Q_{\psi}(x)\leftrightarrow\neg Q_{\sim\psi}(x))$. The reverse implication is often known as the positive shortcut (named in this way because this is a positive formulae when expressed in clausal form). The formal definition justifying the inclusion of these formulae in the translation can be seen in clause 4 of definition 4.1 in [1]. The new formulae are of course added ($\wedge$) to the translated output. The translation with positive shortcuts included is chosen as the default translation because the positive shortcuts act as *guesses* to the successful proof or refutation, and can result in both shorter proofs, and quicker calculations (although this is not always the case [1]). For the translation of many modal axiom combinations, these shortcuts are in fact required to ensure a complete translation (see table 2.5). In other cases, they are optional, but usually desirable.

2.1.5. <u>Mixed translation modes – correspondence properties and schema encoding of axioms.</u> [1]

In [1] several mixed modes of translation are considered, in which axiomatic translation of the modal formula, is combined with axiomatic translation of a subset of the valid modal axioms, while other axioms are represented the correspondence property. These are

KT4B = S5	=	$Corr(T,B) \wedge \prod^{4}$ where $\chi_4 = \chi^{\in}_{\varphi}$	(with shortcuts optional)
KDB	=	$Corr(D) \wedge \prod^{B}$ where $\chi_B = \chi^{\in}_{\varphi}$	(with shortcuts optional)
KD4	=	$Corr(D) \wedge \prod^{4}$ where $\chi_4 = \chi^{\in}_{\varphi}$	(with shortcuts optional)

Potentially such mixed mode translations can take advantage of some beneficial properties of both modes of translation. For example, a mixed translation would be beneficial if the correspondence property produced a faster result under resolution than the axiomatic translation for one modal axiom, and the axiomatic translation a faster result for a different modal axiom.

2.1.6. <u>Multi-modal formulae.</u>

In multimodal formulae, it is possible to distinguish between different instances of the same modal operator. In the notation used in this study, a modality index is attached to each modal operator, and indicates both it's scope and allegiances within the modal formula. Each modality index is associated with a unique accessibility relationship. In the previous uni-modal examples, this modality index has been denoted by r in $\Box(r,p)$ and $\Diamond(r,p)$, and in the translated formulae, R is the corresponding accessibility relationship. The notation supporting multi-modal operators will be illustrated by extending the example, $\Box(r,(\neg\Box(r,p)\wedge\Box(r,q)))$. A possible multimodal interpretation of this is $\Box(r, (\neg\Box(s,p) \wedge \Box(t,q)))$. The basic translation is as seen previously $\prod(\Box((\neg\Box p)\wedge\Box q))$ except that the symbols representing the accessibility relations will be different for each case of the box operator. The sub-formulae are as follows $\{ \Box(r, (\neg\Box(s,p) \wedge \Box(t,q))), \neg\Box(s,p) \wedge \Box(t,q), \Box(s,p), \Box(t,q), p, q \}$. So, using R, S, and T for the accessibility relations corresponding to modality indices r, s, t, the translation is

$\Pi(\Box(r, (\neg\Box(s,p) \wedge \Box(t,q)))) = \exists x Q_{\Box(r,(\neg\Box(s,p)\wedge\Box(t,q)))}(x)$

$\wedge\ \forall x(Q_{\Box(r,(\neg\Box(s,p)\wedge\Box(t,q)))}(x) \rightarrow \forall y(R(x,y)\rightarrow Q_{\neg\Box(s,p)\wedge\Box(t,q)}(y))$ [Def($\Box(r,(\neg\Box(s,p)\wedge\Box(t,q)))$)]

$\wedge\ \forall x(Q_{\Box(r,(\neg\Box(s,p)\wedge\Box(t,q)))}(x) \leftrightarrow \neg Q_{\neg\Box(\neg\Box p\wedge\Box q)}(x))$

$\wedge\ \forall x(Q_{\neg\Box(r,(\neg\Box(s,p)\wedge\Box(t,q)))}(x) \rightarrow \exists y(R(x,y)\wedge Q_{\neg(\neg\Box(s,p)\wedge\Box(t,q))}(y))$

$\wedge\ \forall x(Q_{\neg\Box(s,p)\wedge\Box(t,q)}(x) \rightarrow (Q_{\neg\Box(s,p)}(x)\wedge Q_{\Box(t,q)}(x))$ [Def($\neg\Box(s,p)\wedge\Box(t,q)$)]

$\wedge\ \forall x(Q_{\neg\Box(s,p)\wedge\Box(t,q)}(x) \leftrightarrow \neg Q_{\neg(\neg\Box(s,p)\wedge\Box(t,q))}(x))$

$\wedge\ \forall x(Q_{\neg(\neg\Box(s,p)\wedge\Box(t,q))}(x) \rightarrow (Q_{\Box(s,p)}(x)\vee Q_{\neg\Box(t,q)}(x))$

$\wedge\ \forall x(Q_{\Box(s,p)}(x) \rightarrow \forall y(S(x,y)\rightarrow Q_{p}(y))$ [Def($\Box(s,p)$)]

$\wedge\ \forall x(Q_{\Box(s,p)}(x) \leftrightarrow \neg Q_{\neg\Box(s,p)}(x))$

$\wedge\ \forall x(Q_{\neg\Box(s,p)}(x) \rightarrow \exists y(S(x,y)\wedge Q_{\neg p}(y))$

$\wedge\ \forall x(Q_{\Box(t,q)}(x) \rightarrow \forall y(T(x,y)\rightarrow Q_{q}(y))$ [Def($\Box(t,q)$)]

$\wedge\ \forall x(Q_{\Box(t,q)}(x) \leftrightarrow \neg Q_{\neg\Box(t,q)}(x))$

$\wedge\ \forall x(Q_{\neg\Box(t,q)}(x) \rightarrow \exists y(T(x,y)\wedge Q_{\neg q}(y))$

$\wedge\ \forall x(Q_{p}(x) \leftrightarrow \neg Q_{\neg p}(x)) \wedge \forall x(Q_{q}(x) \leftrightarrow \neg Q_{\neg q}(x))$ [from Def(p) & Def(q)]

The modality indices have been omitted in previous examples because the cluttering is excessive. However, these indices are very important for multi-modal examples, and must always be included since ambiguity can arise. Hence, Def($\Box(s,p)$) is different to Def($\Box(t,p)$), even though in the truncated notation they both appear to be Def($\Box p$).

Differences in the multi-modal translation are particularly important when it comes to applying axioms. For example consider the modal problem, $\Box(r, (\neg\Box(s,p) \wedge \Box(t,q)))$ in axiom T for r, axiom B for s, and axiom D for t. In this case the non-compositional translation is

$\Pi(\Box(r, (\neg\Box(s,p) \wedge \Box(t,q))))$

$\wedge\ \forall x(\neg Q_{\Box(r,(\neg\Box(s,p)\wedge\Box(t,q)))}(x) \vee Q_{\neg\Box(s,p)\wedge\Box(t,q)}(x))$ T,r($\Box(r, (\neg\Box(s,p) \wedge \Box(t,q)))$)

$\wedge\ \forall x(\forall y(\neg S(x,y)\vee\neg Q_{\Box(s,p)}(y)) \vee Q_{p}(x))$ B,s($\Box(s,p)$])

$\wedge\ \forall x(\neg Q_{\Box(t,q)}(x) \vee Q_{\neg\Box(t,\neg q)}(x))$ D,t($\Box(t,q)$])

$\wedge\ Def(\Box(t,\neg q))$

Notice, that the instantiation set for each axiom is effectively *edited* according to the modality index. A further example illustrates this editing and composition in multimodal cases.

Consider the translation of $\neg\Box(r,\neg\Box(a,p)) \wedge \neg\Box(a,\neg\Box(r,\neg p))$ with axiom 5o applied only to modality index a.

The translation is $Def(\neg\Box(r,\neg\Box(a,p)) \wedge \neg\Box(a,\neg\Box(r,\neg p)))$ $\wedge$ $Def(p)$

$\wedge\ Def(\Box(r,\neg\Box(a,p)))$ $\wedge\ Def(\Box(a,\neg\Box(r,\neg p))$

$\wedge\ Def(\Box(a,p))$ $\wedge\ Def(\Box(r,\neg p))$

$\wedge\ 5(\Box(a,\neg\Box(r,\neg p)) \wedge 5(\Box(a, \neg\Box(a,\neg\Box(r,\neg p)))$ $\wedge\ Def(\Box(a,\neg\Box(a,\neg\Box(r,\neg p)))$

$\wedge\ 5(\Box(a,p))$ $\wedge\ 5(\Box(a,\neg\Box(a,p)))$ $\wedge\ Def(\Box(a,\neg\Box(a,p)))$

The behavior of combinations of axioms in the multimodal case is not defined in [1], and is inferred here without proof. When axioms are applied to different modalities, then they do not interact via the instantiation set; that is, the elements in the instantiation set carry an explicit reference to the modality index. When axioms are referring to the same modality, then their interaction is as already described.

2.1.7. <u>Bi-modal axioms.</u>

Bi-modal axioms are very similar to multimodal formulae. The bi-modal axiom is defined in terms of two box operator species, each identified by an independent modality index. The input problem will usually have at least two corresponding modality indices. The bi-modal axioms considered in this study are listed in figure 2.7. The two modality indices in the bi-modal axiom need to be defined in terms of (or mapped to) the two modality indices in the input problem (potentially both can map to the same modality index). During instantiation of the schema for the bi-modal axiom, these two modality indices are treated differently, as the examples below illustrate. Note an alternative notation is used here: $\Box(r,p))$ is written [r]p. The change in notation is arbitrary.

Figure 2.7. Translation of Bi-modal Axioms.

Axiom		Correspondence Property	AxiomaticTranslation		
			Schema Encoding	New Term	Composition Term
CR:	$[r]p \rightarrow [s]p$		$\forall x(\neg Q_{[r]p}(x) \vee Q_{[s]p}(x))$	None	None
CR2:	$[r]p \rightarrow [s][r]p$	$(R_s(x,y) \wedge R_r(y,z)) \rightarrow R_r(x,z)$	$\forall x(\neg Q_{[r]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[r]p}(y)))$	None	[s][r]p
CR3:	$[r]p \rightarrow [r][s]p$	$(R_r(x,y) \wedge R_s(y,z)) \rightarrow R_r(x,z)$	$\forall x(\neg Q_{[r]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[s]p}(y)))$	None	[r][s]p

If an instantiation set {[r]¬[s]p, [r]p, [s]p, [s]¬[r]p, [r]¬[r]p} is considered (there is actually no need to define the original target formula) for modal axioms CR3 and CR2, then there are four possible ways in which the modality indices can be arranged. Naming the modality indices as follows: the "*r*" in the table 2.7 is r_A, and "*s*" is s_A, and the modality indices from target formula are likewise r_T and s_T, then the four possible mappings of the modality indices in the axiom to the modality indices in the target formula are as listed in the table below. The instantiation of the schema clauses for each mapping is also given in the table. It is seen that in effect the first modality index is used to *edit* the instantiation set.

mapping	edited instantiation set	translation for CR3	translation for CR2
$r_A \mapsto r_T$ $s_A \mapsto r_T$	[r]p [r]¬[s]p [r]¬[r]p	$\forall x(\neg Q_{[r]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[r]p}(y)))$ $\forall x(\neg Q_{[r]\neg[s]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[r]\neg[s]p}(y)))$ $\forall x(\neg Q_{[r]\neg[r]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[r]\neg[r]p}(y)))$	$\forall x(\neg Q_{[r]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[r]p}(y)))$ $\forall x(\neg Q_{[r]\neg[s]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[r]\neg[s]p}(y)))$ $\forall x(\neg Q_{[r]\neg[r]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[r]\neg[r]p}(y)))$
$r_A \mapsto r_T$ $s_A \mapsto s_T$	[r]p [r]¬[s]p [r]¬[r]p	$\forall x(\neg Q_{[r]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[s]p}(y)))$ $\forall x(\neg Q_{[r]\neg[s]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[s]\neg[s]p}(y)))$ $\forall x(\neg Q_{[r]\neg[r]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[s]\neg[r]p}(y)))$	$\forall x(\neg Q_{[r]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[r]p}(y)))$ $\forall x(\neg Q_{[r]\neg[s]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[r]\neg[s]p}(y)))$ $\forall x(\neg Q_{[r]\neg[r]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[r]\neg[r]p}(y)))$
$r_A \mapsto s_T$ $s_A \mapsto r_T$	[s]p [s]¬[r]p	$\forall x(\neg Q_{[s]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[r]p}(y)))$ $\forall x(\neg Q_{[s]\neg[r]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[r]\neg[r]p}(y)))$	$\forall x(\neg Q_{[s]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[s]p}(y)))$ $\forall x(\neg Q_{[s]\neg[r]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[s]\neg[r]p}(y)))$
$r_A \mapsto s_T$ $s_A \mapsto s_T$	[s]p [s]¬[r]p	$\forall x(\neg Q_{[s]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[s]p}(y)))$ $\forall x(\neg Q_{[s]\neg[r]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[s]\neg[r]p}(y)))$	$\forall x(\neg Q_{[s]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[s]p}(y)))$ $\forall x(\neg Q_{[s]\neg[r]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[s]\neg[r]p}(y)))$

No proof of this material is offered. It represents a logical extension of the axiomatic translation that is suggested in [1]. A derivation is given in the results section.

Note that while multi-modal target formulae and bi-modal axioms are considered in this study, n-ary axioms (referring to more than one modal formula) are not considered. An example would be axiom H. (Hintikka corresponding to modal formula $\neg\Box(\Box p \to q) \to \Box(\Box q \to p)$, with a correspondence property $(R(x,y) \vee R(x,z)) \to (R(y,z) \vee R(z,y))$; and schema encoding $\forall x(\ \forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y) \vee Q_q(y)) \ \vee \ \forall y(\neg R(x,y) \vee \neg Q_{\Box q}(y) \vee Q_p(y))\ )$.

2.2. Resolution.

The authors of [1] go to considerable effort to demonstrate completeness of the axiomatic translation under resolution for the cases described above. (If there is a proof by axiomatic translation for every theorem, then it is complete). Soundness is proved in [1] much more easily. (If every theorem proven using the translation is in fact valid, then it is sound). As a result it is shown in [1] that for the both axiomatic and mixed modes of translation, that (i) a modal formula is (un)satisfiable in the K if and only if the translation of the formulae is (un)satisfiable in first-order logic, and (ii) a modal formula is (un)satisfiable in the $K\alpha_1 \ldots \alpha_n$ (where $\alpha_1 \ldots \alpha_n$ is a sequence of modal axioms) if and only if the translation of the formulae in this sequence of modal axioms is (un)satisfiable in first-order logic. This is the basis on which the satisfiablity of modal target formulae is determined in extended-SPASS.

There are other points that are worth noting. It is shown in [1] that the axiomatic translation of modal target formulae with different modal axioms, yields formulae that are known to be *decidable* under first order resolution. This is because only formulae composed of fragments known to be decidable can be formed by the translation. (These decidable fragments are members of the classes of formulae known as guarded fragments (GF^2, [1, 23]) or DL* fragments [1, 22]). This is an important result. There is no direct means by which formulae with 'triangular' properties, of the types embodied in the modal axioms 4, 5, and alt_1 can be always expressed in these (or other) decidable fragments. This then is the major advantage of the axiomatic translation. It is expected that many problems that are impossible to solve in axioms 4, 5 and alt_1, will be soluble under axiomatic translation. Indeed this is the finding in [1] and in the results section of this study. Unfortunately, it will also be seen that for higher order axioms 4^κ, 5^κ and $alt1^{\kappa\kappa}$, there are additional problems that are not addressed by the axiomatic translation.