The Axiomatic Translation of Modal Logics.

Dr. K. John Smith
Dr. Renate S. Schmidt (supervisor).

Based on work described in
Schmidt, R. A. and Hustadt, U. (2007). The Axiomatic Translation Principle for Modal Logic. Transactions on Computational Logic. 8(4).

Overview

MODAL LOGIC

$\neg$ $\rightarrow$

$\wedge$ $\vee$

$\leftarrow$ $\leftrightarrow$

$\Box$ $\Diamond$

$\top$ $\bot$

FIRST ORDER LOGIC (FOL)

$\forall x$ $\exists y$

$R(x,y)$ $Q(x)$

$\neg$ $\rightarrow$

$\leftarrow$ $\leftrightarrow$

$\wedge$ $\vee$

$\top$ $\bot$

SPASS3.0

RESOLUTION PROVER

(refutation)

- **Satisfiable** (model)
- **Unsatisfiable** (contradiction)
- **Unknown** (Ran out of of Execution Time)

- Classical Translation & Correspondence Properties
- Axiomatic Translation
- Mixtures

extended - SPASS

Kripke semantics (or Possible World semantics)

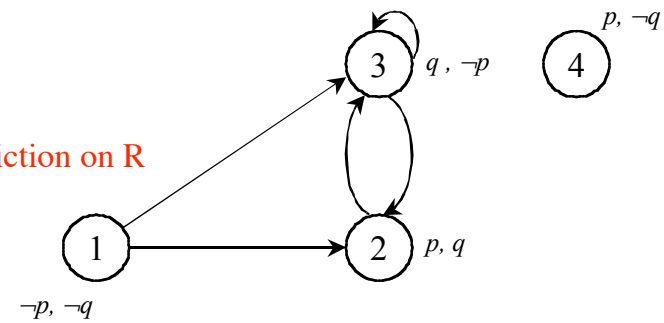
- Kripke frame F is an ordered pair (W, R)
  - W set of Possible Worlds
  - **R binary Accessibility Relation** connecting some worlds ($R \subseteq (W x W)$)
- model M = (W, R, I), where I is interpretation (or valuation)
  - Example: frame F = {W={1,2,3,4}, R={(1,2), (1,3), (2,3), (3,2), (3,3)}}
    particular interpretation I = {(p, {2,4}), (q, {2,3})}

Node == W

Edge == R

Axioms are a restriction on R

Label == I



so, worlds 2 and 3 are successor worlds of world 3
and, p is true in world 2 (so, $2 \in I(p)$), and false in world 1

Consider a particular world

- box operator ($\Box\phi$) true if formula (ϕ) is true in *all* immediate successor worlds
- diamond operator ($\Diamond\phi$) true if formula is true in *at least one* successor world

Example $\Box\Box p$ is false at world 1 in model M
p is false for transitions 1 to 3 to 3; p is false for transitions 1 to 2 to 3

satisfiable = true for some world/model
valid = true for every world/model

Hand-crafted proofs in specific situations are possible

- e.g. Model Checking Algorithm (not shown)

But what about **automated proofs**?

Classical Translation of Modal Formula

(not implemented here)

$\pi(p, x) = Q_p(x)$

$\pi(\bot, x) = \bot$

$\pi(\top, x) = \top$

$\pi(\neg\phi, x) = \neg\pi(\phi, x)$

$\pi(\psi * \phi, x) = \pi(\psi, x) * \pi(\phi, x)$

where $* \in \{\rightarrow, \leftrightarrow, \vee, \wedge\}$

$\pi(\Box\psi, x) = \forall y(R(x, y) \rightarrow \pi(\psi, y))$

$\pi(\Diamond\psi, x) = \exists y(R(x, y) \wedge \pi(\Box\psi, y))$

$\Diamond\psi = \neg\Box\neg\psi$

p is propositional variable

ψ, ϕ are modal formulae

x, y are unique FOL variables

- Translate a modal formula by *exhaustive* application of the above transformations
- **ϕ is valid in K iff**

  $\forall x(\pi(\phi, x))$ is valid in FOL

That is **$\exists x(\pi(\neg\phi, x))$ is unsatisfiable in FOL**

Correspondence Properties

Axiom	Modal Axiom	Property of R	**Correspondence Property** (*Corr*)
T	$\Box p \rightarrow p$	Reflexive	$\forall x\, R(x,x)$
D	$\Box p \rightarrow \Diamond p$	Serial	$\forall x \exists y\, R(x,y)$
B	$p \rightarrow \Box\Diamond p$	Symmetry	$\forall xy\, (R(x,y) \rightarrow R(y,x))$
4	$\Box p \rightarrow \Box\Box p$	Transitive	$\forall xyz\, ((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$
5	$\Diamond p \rightarrow \Box\Diamond p$	Euclidean	$\forall xyz\, ((R(x,y) \wedge R(x,z)) \rightarrow R(y,z))$
alt_1	$\Diamond p \rightarrow \Box p$	Functionality	$\forall xyz\, ((R(x,y) \wedge R(x,z)) \rightarrow (y \approx z))$

ETC…

modal formula, ϕ, is valid in K.Axiom iff

$Corr_{Axiom} \wedge \exists x(\pi(\neg\phi, x))$ is unsatisfiable in FOL

<u>Scott-Lemmon Algorithm</u> (1975):

Modal Formula …

$\Diamond^h\Box^i\phi \rightarrow \Box^j\Diamond^k\phi$

where values h=0, i=1, j=2, k=3 in $\Box\phi \rightarrow \Box\Box\Diamond\Diamond\Diamond\phi$

Correspondence Property …

$\forall vyz(R^h(v,y) \wedge R^j(v,z)) \rightarrow \exists x(R^i(y,x) \wedge R^k(z,x))$

where R^0 means equivalence

$R^2(x,y)$ means $R(x,z) \wedge R(z,y)$

etc

Axiom 5 is $\Diamond\Box\phi \rightarrow \Box\phi$ with values h=1, i=1, j=1, k=0, and a correspondence property

$\forall vyz\, (R^1(v,y) \wedge R^1(v,z)) \rightarrow \exists x(R^1(y,x) \wedge R^0(z,x))$

$\forall vyz\, (R(v,y) \wedge R(v,z)) \rightarrow \exists x(R(y,x) \wedge (z=x))$

$\forall vyz\, (R(v,y) \wedge R(v,z)) \rightarrow R(y,z)$ Euclidean

<u>Others</u>… SCAN, handcrafted

Implemented in extended-SPASS

- extended-SPASS does not implement classical translation of modal axiom, but does implement correspondence properties of axiom

For example, $\Box((\neg\Box(r,p)) \wedge \Box(r,q))$ in KB_c is simply

$\prod(\Box((\neg\Box(r,p)) \wedge \Box(r,q)))$ [Translation of modal formula]

$\wedge\ \forall xy(R(x,y) \rightarrow R(y,x))$ [*Corr*(B)]

Classical translation of some axioms of modal logic

Axiom	Classical Translation
T reflexive	$\forall x(R(x,x))$
D serial	$\forall x\exists y(R(x,y))$
B symmetry	$\forall xy(R(x,y) \rightarrow R(y,x))$
4 transitive	$\forall xyz((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$
5 Euclidean	$\forall xyz(R(x,y) \wedge R(x,z) \rightarrow R(y,z))$
alt_1 functional	$\forall xyz(R(x,y) \wedge R(x,z) \rightarrow (y\approx z))$
4^K 4^2 4^3	$\forall xy(R^{K+1}(x,y) \rightarrow R(x,y))$ $\forall xuyz((R(x,u) \wedge R(u,y) \wedge R(y,z)) \rightarrow R(x,z))$ $\forall xuvyz((R(x,u) \wedge R(u,v) \wedge R(v,y) \wedge R(y,z)) \rightarrow R(x,z))$
5^K 5^2 5^3	$\forall xyz(R^K(x,y) \wedge R(x,z) \rightarrow R(y,z))$ $\forall xuyz((R(x,u) \wedge R(u,y) \wedge R(x,z)) \rightarrow R(y,z))$ $\forall xuvyz((R(x,u) \wedge R(u,v) \wedge R(v,y) \wedge R(x,z)) \rightarrow R(y,z))$
$alt_1^{K1,K2}$ $alt_1^{1,1}$ $alt_1^{1,2}$ $alt_1^{2,1}$ $alt_1^{2,2}$	$\forall xyz(R^{K1+1}(x,y) \wedge R^{K2+1}(x,z) \rightarrow (y\approx z))$ $\forall xyz((\exists u(R(x,u) \wedge R(u,y)) \wedge \exists v(R(x,v) \wedge R(v,z))) \rightarrow (y\approx z))$ $\forall xyz((\exists u(R(x,u) \wedge R(u,y)) \wedge \exists vw(R(x,v) \wedge R(v,w) \wedge R(w,z))) \rightarrow (y\approx z))$ $\forall xyz((\exists uw(R(x,u) \wedge R(u,w) \wedge R(w,y)) \wedge \exists v(R(x,v) \wedge R(v,z))) \rightarrow (y\approx z))$ $\forall xyz((\exists uw(R(x,u) \wedge R(u,w) \wedge R(w,y)) \wedge \exists vg(R(x,v) \wedge R(v,g) \wedge R(g,z))) \rightarrow (y\approx z))$

extended-SPASS Syntax

- *Axiom combination* $K5^3c$ *is defined by flags*

```
'-EMLAxiom=1
—EMLClassLog5K=1 —EMLClassLogFac5=3'.
```

- *Reports …*

```
Requested Composition ..... K,5Kc[3]
```

```
list_of_settings(SPASS).
{*
set_flag(EMLAxiom,1).
set_axiomflag([r,Bc],[r,5c],[r,Tc]).
*}
```

```
…
list_of_symbols.
      predicates[(Rr,2),(r,0),(p,0)].
      translpairs[(r,Rr)].
end_of_list.

list_of_special_formulae(axioms,eml).
 prop_formula(noAxiom(
   forall([x,y],equiv(Rr(x,y),equal(x,y))) %Axiom TR
 )).
 prop_formula(
      and(box(r,p),dia(r,true))
 ).
end_of_list.
…
```

- *slf(modality index, formula)* $\Diamond^h\Box^j p \rightarrow \Box^j\Diamond^k p$

```
list_of_special_formulae(axioms,eml).
prop_formula(
   not(implies(not(box(r,not(box(r,p))))
   ,box(r,p)))
).
prop_formula(slf(
 r,implies(dia(r,box(r,p)),box(r,dia(r,p)))
)).
end_of_list.
```

Scott-Lemmon translation (correspondence properties) for selected axioms

with results for translation of formula 5 in each axiom (500 seconds execution time; C = Completion (Satisfiable); P = Proof (Unsatisfiable); X = ran out of execution time)

$\Diamond^h\Box^i\phi \rightarrow \Box^j\Diamond^k\phi$

SLF index	Axiom		Term created	Result
0000	None	$\Diamond^0\Box^0p\rightarrow\Box^0\Diamond^0p$	TRUE	C
1000		$\Diamond^1\Box^0p\rightarrow\Box^0\Diamond^0p$	$\forall[W,V](R_r(V,W)\rightarrow(W\approx V))$	P
0100	T	$\Diamond^0\Box^1p\rightarrow\Box^0\Diamond^0p$	$\forall[W,V]((W\approx V)\rightarrow R_r(W,V))$	C
0010		$\Diamond^0\Box^0p\rightarrow\Box^1\Diamond^0p$	$\forall[W,V](R_r(V,W)\rightarrow(W\approx V))$	P
0001		$\Diamond^0\Box^0p\rightarrow\Box^0\Diamond^1p$	$\forall[W,V]((W\approx V)\rightarrow R_r(W,V))$	C
1100		$\Diamond^1\Box^1p\rightarrow\Box^0\Diamond^0p$	$\forall[W,V](R_r(V,W)\rightarrow R_r(W,V))$**	C
1010	Alt_1	$\Diamond^1\Box^0p\rightarrow\Box^1\Diamond^0p$	$\forall[V,W,U]((R_r(U,V)\wedge R_r(U,W))\rightarrow(W\approx V))$	X
1001		$\Diamond^1\Box^0p\rightarrow\Box^0\Diamond^1p$	$\forall[W,V](R_r(V,W)\rightarrow R_r(W,V))$**	C
0110		$\Diamond^0\Box^1p\rightarrow\Box^1\Diamond^0p$	$\forall[W,V](R_r(V,W)\rightarrow R_r(W,V))$**	C
0101	D	$\Diamond^0\Box^1p\rightarrow\Box^0\Diamond^1p$	$\forall[W,V]((W\approx V)\rightarrow\exists[X](R_r(V,X)\wedge R_r(W,X)))$	C
0011	B	$\Diamond^0\Box^0p\rightarrow\Box^1\Diamond^1p$	$\forall[W,V](R_r(V,W)\rightarrow R_r(W,V))$**	C
0111		$\Diamond^0\Box^1p\rightarrow\Box^1\Diamond^1p$	$\forall[W,V](R_r(V,W)\rightarrow\exists[X](R_r(V,X)\wedge R_r(W,X)))$	C
1101		$\Diamond^1\Box^1p\rightarrow\Box^0\Diamond^1p$	$\forall[W,V](R_r(V,W)\rightarrow\exists[X](R_r(V,X)\wedge R_r(W,X)))$	C
1011	5	$\Diamond^1\Box^0p\rightarrow\Box^1\Diamond^1p$	$\forall[V,W,U]((R_r(U,V)\wedge R_r(U,W))\rightarrow R_r(W,V))$	P
1110		$\Diamond^1\Box^1p\rightarrow\Box^1\Diamond^0p$	$\forall[V,W,U]((R_r(U,V)\wedge R_r(U,W))\rightarrow R_r(W,V))$	P
1111	G	$\Diamond^1\Box^1p\rightarrow\Box^1\Diamond^1p$	$\forall[V,W,U]((R_r(U,V)\wedge R_r(U,W))\rightarrow\exists[X](R_r(V,X)\wedge R_r(W,X)))$	X
0120	4	$\Diamond^0\Box^1p\rightarrow\Box^2\Diamond^0p$	$\forall[V,W,Y]((R_r(Y,W)\wedge R_r(V,Y))\rightarrow R_r(W,V))$	X
0210	DEN	$\Diamond^0\Box^2p\rightarrow\Box^1\Diamond^0p$	$\forall[Y,W,V](R_r(V,W)\rightarrow(R_r(Y,V)\wedge R_r(W,Y)))$	P
1002		$\Diamond^1\Box^0p\rightarrow\Box^0\Diamond^2p$	$\forall[Y,W,V](R_r(V,W)\rightarrow(R_r(Y,V)\wedge R_r(W,Y)))$	P
1120	DBBB	$\Diamond^1\Box^1p\rightarrow\Box^2\Diamond^0p$	$\forall[V,U,W,Y]((R_r(U,V)\wedge(R_r(Y,W)\wedge R_r(U,Y)))\rightarrow R_r(W,V))$	X
2100	B^2	$\Diamond^2\Box^1p\rightarrow\Box^0\Diamond^0p$	$\forall[W,V,Y]((R_r(Y,V)\wedge R_r(W,Y))\rightarrow R_r(W,V))$	X
0002		$\Diamond^0\Box^0p\rightarrow\Box^0\Diamond^2p$	$\forall[Y,W,V]((W\approx V)\rightarrow(R_r(Y,V)\wedge R_r(W,Y)))$	P
0020		$\Diamond^0\Box^0p\rightarrow\Box^2\Diamond^0p$	$\forall[V,W,Y]((R_r(Y,W)\wedge R_r(V,Y))\rightarrow(W\approx V))$	X
2200		$\Diamond^2\Box^2p\rightarrow\Box^0\Diamond^0p$	$\forall[Z,W,V,Y]((R_r(Y,V)\wedge R_r(W,Y))\rightarrow(R_r(Z,V)\wedge R_r(W,Z)))$	C
2220		$\Diamond^2\Box^2p\rightarrow\Box^2\Diamond^0p$	$\forall[X_1,V,Z,U,W,Y](((R_r(Z,V)\wedge R_r(U,Z))\wedge(R_r(Y,W)\wedge R_r(U,Y)))\rightarrow(R_r(X_1,V)\wedge R_r(W,X_1)))$	C
2020		$\Diamond^2\Box^0p\rightarrow\Box^2\Diamond^0p$	$\forall[V,Z,U,W,Y](((R_r(Z,V)\wedge R_r(U,Z))\wedge(R_r(Y,W)\wedge R_r(U,Y)))\rightarrow(W\approx V))$	X
0202		$\Diamond^0\Box^2p\rightarrow\Box^0\Diamond^2p$	$\forall[Z,Y,W,V]((W\approx V)\rightarrow\exists[X]((R_r(Z,X)\wedge R_r(V,Z))\wedge(R_r(Y,X)\wedge R_r(W,Y))))$	P
2022		$\Diamond^2\Box^0p\rightarrow\Box^2\Diamond^2p$	$\forall[X_1,V,Z,U,W,Y](((R_r(Z,V)\wedge R_r(U,Z))\wedge(R_r(Y,W)\wedge R_r(U,Y)))\rightarrow(R_r(X_1,V)\wedge R_r(W,X_1)))$	C
2202		$\Diamond^2\Box^2p\rightarrow\Box^0\Diamond^2p$	$\forall[X_1,Z,W,V,Y]((R_r(Y,V)\wedge R_r(W,Y))\rightarrow\exists[X]((R_r(X_1,X)\wedge R_r(V,X_1))\wedge(R_r(Z,X)\wedge R_r(W,Z)))$)	C
2000		$\Diamond^2\Box^0p\rightarrow\Box^0\Diamond^0p$	$\forall[W,V,Y]((R_r(Y,V)\wedge R_r(W,Y))\rightarrow(W\approx V))$	X
2222		$\Diamond^2\Box^2p\rightarrow\Box^2\Diamond^2p$	$\forall[X_2,X_1,V,Z,U,W,Y](((R_r(Z,V)\wedge R_r(U,Z))\wedge(R_r(Y,W)\wedge R_r(U,Y)))\rightarrow\exists[X]((R_r(X_2,X)\wedge R_r(V,X_2))\wedge(R_r(X_1,X)\wedge R_r(W,X_1))))$	C
0022		$\Diamond^0\Box^0p\rightarrow\Box^2\Diamond^2p$	$\forall[Z,V,W,Y]((R_r(Y,W)\wedge R_r(V,Y))\rightarrow(R_r(Z,V)\wedge R_r(W,Z)))$	C
0222		$\Diamond^0\Box^2p\rightarrow\Box^2\Diamond^2p$	$\forall[X_1,Z,V,W,Y]((R_r(Y,W)\wedge R_r(V,Y))\rightarrow\exists[X]((R_r(X_1,X)\wedge R_r(V,X_1))\wedge(R_r(Z,X)\wedge R_r(W,Z)))$	C
3333		$\Diamond^3\Box^3p\rightarrow\Box^3\Diamond^3p$	$\forall[X_6,X_5,X_4,X_3,V,X_2,X_1,U,W,Z,Y](((R_r(X_2,V)\wedge(R_r(X_1,X_2)\wedge R_r(U,X_1)))\wedge(R_r(Z,W)\wedge(R_r(Y,Z)\wedge R_r(U,Y))))\rightarrow\exists[X]((R_r(X_6,X)\wedge(R_r(X_5,X_6)\wedge R_r(V,X_5)))\wedge(R_r(X_4,X)\wedge(R_r(X_3,X_4)\wedge R_r(W,X_3)))))$	C
3100	B^3	$\Diamond^3\Box^1p\rightarrow\Box^0\Diamond^0p$	$\forall[W,V,Z,Y]((R_r(Z,V)\wedge(R_r(Y,Z)\wedge R_r(W,Y)))\rightarrow R_r(W,V))$	X

** Several formulations give rise to the same axiom under the Scott-Lemmon algorithm

Results and Execution times for Translation of 119 problems with modal axioms translated as Correspondence Properties

200 seconds execution time; Completion (Satisfiable); Proof (Unsatisfiable)

Axiom	Total No. Problems Solved (failed)	Execution Time in msec			Proof No. Problems Solved	Execution Time in msec			Completion No. Problems Solved	Execution Time in msec		
		Median	Mean±SD	Max		Median	Mean±SD	Max		Median	Mean±SD	Max
T_c	119(0)	20	32±29	210	15	20	32±51	210	104	20	32±25	130
B_c	119(0)	30	118±327	2890	21	30	46±46	180	98	30	133±358	2890
D_c	119(0)	20	28±23	140	8	10	26±38	120	111	20	28±22	140
4_c	25(94)	70	622±1603	7970	25	70	622±1603	7970	0			
5_c	48(71)	115	628±1249	5320	48	115	628±1249	5320	0			
Alt_{1c}	18(101)	35	55±50	180	18	35	55±50	180	0			
4^2_c	11(108)	830	21267±44005	128870	11	830	21267±44005	128870	0			
4^3_c	5(114)	70	21702±32243	72050	5	70	21702±32243	72050	0			
5^2_c	14(105)	2125	29384±48904	149420	14	2125	29384±48904	149420	0			
5^3_c	3(116)	10	37±46	90	3	10	37±46	90	0			
$Alt_1^{1,1}{}_c$	8(111)	8945	28794±42380	114670	8	8945	28794±42380	114670	0			
$Alt_1^{2,1}{}_c$	5(114)	130	656±878	1980	5	130	656±878	1980	0			
$Alt_1^{1,2}{}_c$	3(116)	20	53±67	130	3	20	53±67	130	0			
$Alt_1^{2,2}{}_c$	3(116)	20	63±84	160	3	20	63±84	160	0			
T_c4_c	35(84)	110	12239±33058	151120	35	110	12239±33058	151120	0			
T_cB_c	119(0)	30	84±176	1170	30	20	47±70	340	89	30	97±198	1170
D_cB_c	119(0)	30	106±336	3060	25	20	36±43	220	94	30	125±375	3060
D_c4_c	42(77)	690	11701±30755	137360	28	1705	17270±36602	137360	14	355	562±520	1600
4_cB_c	69(50)	140	860±2888	21490	69	140	860±2888	21490	0			
5_cB_c	71(48)	90	4458±24466	191040	71	90	4458±24466	191040	0			
T_c5_c	77(42)	140	683±2977	25940	77	140	683±2977	25940	0			
$T_c4_cB_c$	75(44)	190	2458±16494	143050	68	115	2583±17325	143050	7	690	1237±1377	3920
$D_c4_cB_c$	57(62)	320	6125±21915	148000	57	320	6125±21915	148000	0			

Difficult problems are $4, 5, Alt_1$ ($4^K, 5^K, Alt_1^{KK}$)

Axiom	Property of R	**Correspondence Property** (*Corr*)
4	Transitive	$\forall xyz\ ((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$
5	Euclidean	$\forall xyz\ ((R(x,y) \wedge R(x,z)) \rightarrow R(y,z))$
alt_1	Functionality	$\forall xyz\ ((R(x,y) \wedge R(x,z)) \rightarrow (y \approx z))$

Why is this not good enough?

- For some axioms [T, B, D] problems are solved quickly - great!
- But, many modal axioms define problems which run out of execution time

- frequently the problem simply increases in size as resolution proceeds with no solution found
- for ordered resolution there is no single maximal literal

- This is true for the 'triangular axioms' [4, alt_1, 5], and also for axioms with larger 'chains' of predicates [4^K, alt_1^{KK}, 5^K]

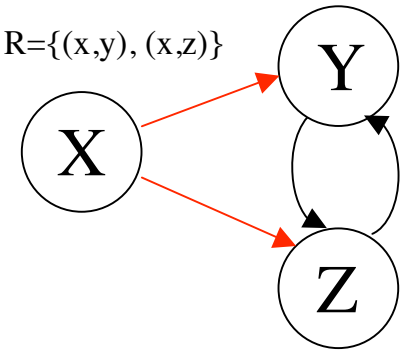
- Not every modal axiom has a FOL correspondence property

example axiom M $\Box\Diamond p \rightarrow \Diamond\Box p$

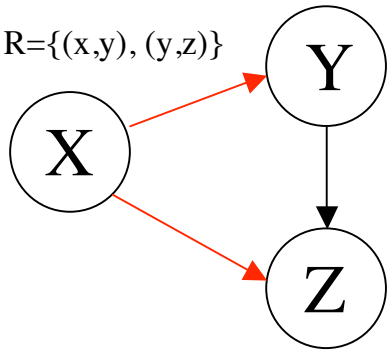
Note: Some properties have no modal equivalent, example asymmetry, irreflexive, antisymmetry, …

Axiom 5. Euclidean $\Diamond p \rightarrow \Box\Diamond p$	Axiom 4. Transitive $\Box p \rightarrow \Box\Box p$
$\forall xyz\ ((R(x,y) \wedge R(x,z)) \rightarrow R(y,z))$	$\forall xyz\ ((R(x,y) \wedge R(y,z)) \rightarrow R(x,z))$

given R={(x,y), (x,z)}



given R={(x,y), (y,z)}



Axiomatic Translation of a modal formula.

Principle: Axiomatic Translation is extension of classical translation, where translation *stops before the final box operator eliminated*, so both p and □p are regarded as *ground terms*, leading to predicate terms like $Q_p(x)$ and $Q_{\Box p}(x)$ ("p and □p hold at world x")

$$\prod(\varphi) = \exists x\, Q_\varphi(x) \wedge \bigwedge\{Def(\psi) \mid \psi \in Sf(\varphi)\}$$

- $Sf(\varphi)$ denotes the set of all sub-formulae of φ
- Q_φ is a predicate symbol uniquely associated with modal formula φ
- Sub-formulae are processed according to definition of $Def(\psi)$

$$\begin{aligned} Def(\psi) \quad &= \quad \forall x(Q_\psi(x) \rightarrow \pi(\psi, x)) \\ \wedge \quad &\quad \forall x(Q_\psi(x) \leftrightarrow \neg Q_{\sim\psi}(x)) \qquad [\forall x(Q_\psi(x) \rightarrow \neg Q_{\sim\psi}(x)) \wedge \forall x(Q_\psi(x) \leftarrow \neg Q_{\sim\psi}(x))] \\ \wedge \quad &\quad \forall x(Q_{\sim\psi}(x) \rightarrow \pi(\sim\psi, x)) \end{aligned}$$

Note this definition includes the reverse implication for the second term

where $\pi(\bot, x) = \bot$ $\quad \pi(p, x) = \top$ $\quad \pi(\neg p, x) = \neg Q_p(x)$

$\pi(\psi \wedge \phi, x) = Q_\psi(x) \wedge Q_\phi(x)$ $\quad \pi(\neg(\psi \wedge \phi), x) = Q_{\sim\psi}(x) \vee Q_{\sim\phi}(x)$

$\pi(\Box\psi, x) = \forall y(R(x, y) \rightarrow Q_\psi(y))$

$\pi(\neg\Box\psi, x) = \exists y(R(x, y) \wedge Q_{\sim\psi}(y))$

$\pi(\psi \wedge \phi \wedge \varphi, x) = Q_\psi(x) \wedge Q_\phi(x) \wedge Q_\varphi(x)$

$\pi(\neg(\psi \wedge \phi \wedge \varphi), x) = Q_{\sim\psi}(x) \vee Q_{\sim\phi}(x) \vee Q_{\sim\varphi}(x)$

Various optimizations, for example

- Translation of a predicate sub-formula
  $\forall x(Q_\psi(x) \leftrightarrow \neg Q_{\sim\psi}(x))$
- Sub-formulae with a leading ¬ are neglected
- Sub-formulae with ∧ can be re-ordered to produce equivalent terms

Shortcuts

- *May be needed for completeness*
- expression $\forall x(Q_\psi(x) \rightarrow \neg Q_{\sim\psi}(x))$ extended by conjugation with $\forall x(Q_\psi(x) \leftarrow \neg Q_{\sim\psi}(x))$ to $\forall x(Q_\psi(x) \leftrightarrow \neg Q_{\sim\psi}(x))$
- In many cases these shortcuts are optional for completene
- Positive shortcuts act as guesses to the successful proof or refutation, often resulting in shorter / quicker proofs

Example of *Instantiation Set* for this modal formula:

↓ ↓ ↓ ↓ ↓ ↓ ↓

□ ((¬ □ p) ∧ □ q)

Listing the sub-formulae gives

1. □((¬□p)∧□q)
2. ¬□p∧□q
3. ¬□p
4. □p
5. p
6. □q
7. q

Axiomatic Translation of Modal Formula in axiom K

An Example

$$\Pi(\Box(\neg\Box p \wedge \Box q)) = \exists x Q_{\Box(\neg\Box p\wedge\Box q)}(x)$$

$$\begin{aligned}
&\wedge\ \forall x(Q_{\Box(\neg\Box p\wedge\Box q)}(x) \rightarrow \forall y(R(x,y) \rightarrow Q_{\neg\Box p\wedge\Box q}(y)))\\
&\wedge\ \forall x(Q_{\Box(\neg\Box p\wedge\Box q)}(x) \leftrightarrow \neg Q_{\neg\Box(\neg\Box p\wedge\Box q)}(x))\\
&\wedge\ \forall x(Q_{\neg\Box(\neg\Box p\wedge\Box q)}(x) \rightarrow \exists y(R(x,y) \wedge Q_{\neg(\neg\Box p\wedge\Box q)}(y)))\\
&\wedge\ \forall x(Q_{\neg\Box p\wedge\Box q}(x) \rightarrow (Q_{\neg\Box p}(x) \wedge Q_{\Box q}(x)))\\
&\wedge\ \forall x(Q_{\neg\Box p\wedge\Box q}(x) \leftrightarrow \neg Q_{\neg(\neg\Box p\wedge\Box q)}(x))\\
&\wedge\ \forall x(Q_{\neg(\neg\Box p\wedge\Box q)}(x) \rightarrow (Q_{\Box p}(x) \vee Q_{\neg\Box q}(x)))\\
&\wedge\ \forall x(Q_{\Box p}(x) \rightarrow \forall y(R(x,y) \rightarrow Q_{p}(y)))\\
&\wedge\ \forall x(Q_{\Box p}(x) \leftrightarrow \neg Q_{\neg\Box p}(x))\\
&\wedge\ \forall x(Q_{\neg\Box p}(x) \rightarrow \exists y(R(x,y) \wedge Q_{\neg p}(y)))\\
&\wedge\ \forall x(Q_{\Box q}(x) \rightarrow \forall y(R(x,y) \rightarrow Q_{q}(y)))\\
&\wedge\ \forall x(Q_{\Box q}(x) \leftrightarrow \neg Q_{\neg\Box q}(x))\\
&\wedge\ \forall x(Q_{\neg\Box q}(x) \rightarrow \exists y(R(x,y) \wedge Q_{\neg q}(y)))\\
&\wedge\ \forall x(Q_{p}(x) \leftrightarrow \neg Q_{\neg p}(x))\\
&\wedge\ \forall x(Q_{q}(x) \leftrightarrow \neg Q_{\neg q}(x))
\end{aligned}$$

MANY terms are created

Standardization of modal formula prior to translation

leaving only $\wedge\ \neg\ \Box\ \ \bot$

Remove Trivial Terms:

$\top \vee p = \top$	$\bot \vee p = p$
$\top \wedge p = p$	$\bot \wedge p = \bot$
$\bot \rightarrow p = \top$	$\top \rightarrow p = p$
$p \rightarrow \bot = \neg p$	$p \rightarrow \top = \top$
$\bot \leftarrow p = \neg p$	$\top \leftarrow p = \top$
$p \leftarrow \bot = \top$	$p \leftarrow \top = p$
$\bot \leftrightarrow p = \neg p$	$\top \leftrightarrow p = p$
$p \leftrightarrow \bot = \neg p$	$p \leftrightarrow \top = p$
$\Box\bot = \bot$	$\Box\top = \top$
$\Diamond\bot = \bot$	$\Diamond\top = \top$

Eliminate Diamond: $\Diamond p = \neg\Box\neg p$

Eliminate True: $\top = \neg\bot$

Eliminate Bi-implication: $(p \leftrightarrow q) = (p \rightarrow q) \wedge (p \leftarrow q)$

Eliminate Implied: $(p \leftarrow q) = (q \rightarrow p)$

Eliminate Implies: $(p \rightarrow q) = (\neg p \vee q)$

Eliminate Or: $(a \vee b) = \neg(\neg a \wedge \neg b)$; $(a \vee b \vee c) = \neg(\neg a \wedge \neg b \wedge \neg c)$; …

Eliminate Nested $\wedge$ / Expand $\Box$ over $\wedge$: $\wedge(p, \wedge(a,b), q, p) = \wedge(p,a,b,q)$

$\Box(p \wedge q) = \Box p \wedge \Box q$

Eliminate Double Negative: $\neg\neg p = p$

Axiomatic Translation of Modal Axioms (1)

Axiomatic Translation of common modal axioms

Axiom	Translation of $\Box(r,p)$	New symbols
T reflexive	$\forall x(\neg Q_{\Box r,p}(x) \vee Q_{r,p}(x))$	None
D serial	$\forall x(\neg Q_{\Box r,p}(x) \vee Q_{\neg\Box r,\neg p}(x))$	$\Box r,\neg p$
4 transitive	$\mathbf{\forall x(\neg Q_{\Box r,p}(x) \vee \forall y(\neg R(x,y) \vee Q_{\Box r,p}(y)))}$	None
5 euclidean	$\mathbf{\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box r,p}(y)) \vee Q_{\Box r,p}(x))}$	None
B symmetry	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box r,p}(y)) \vee Q_{r,p}(x))$	None
Alt_1 functional	$\forall x(\neg Q_{\neg\Box r,p}(x) \vee Q_{\Box r,\neg p}(x))$	$\Box r,\neg p$
4^K 4^2 4^3	$\forall x(\neg Q_{\Box r,p}(x) \vee \forall y(\neg R^K(x,y) \vee Q_{\Box r,p}(y))$ $\forall x(\neg Q_{\Box r,p}(x) \vee \forall yz(\neg R(x,y) \vee \neg R(y,z) \vee Q_{\Box r,p}(z)))$ $\forall x(\neg Q_{\Box r,p}(x) \vee \forall yzu(\neg R(x,y) \vee \neg R(y,z) \vee \neg R(z,u) \vee Q_{\Box r,p}(u)))$	None
5^K $\mathbf{5^2}$ $\mathbf{5^3}$	$\forall x(Q_{\Box r,p}(x) \vee \forall y(\neg R^K(x,y) \vee \neg Q_{\Box r,p}(y)))$ $\mathbf{\forall x(Q_{\Box r,p}(x) \vee \forall zy(\neg R(x,z) \vee \neg R(z,y) \vee \neg Q_{\Box r,p}(y)))}$ $\mathbf{\forall x(Q_{\Box r,p}(x) \vee \forall zuy(\neg R(x,z) \vee \neg R(z,u) \vee \neg R(u,y) \vee \neg Q_{\Box r,p}(y)))}$	None
alt_1^{K1K2} $alt_1^{1,1}$ $alt_1^{1,2}$ $alt_1^{2,1}$ $alt_1^{2,2}$	$\forall xyz(\neg R^{K1}(x,y) \vee \neg R^{K2}(x,z) \vee \neg Q_{\neg\Box r,p}(y) \vee Q_{\Box r,\neg p}(z))$ $\forall x(\forall y[\neg R(x,y) \vee \neg Q_{\neg\Box r,p}(y)] \vee \forall z[\neg R(x,z) \vee Q_{\Box r,\neg p}(z)])$ $\forall x(\forall y[\neg R(x,y) \vee \neg Q_{\neg\Box r,p}(y)] \vee \forall uz[\neg R(x,u) \vee \neg R(u,z) \vee Q_{\Box r,\neg p}(z)])$ $\forall x(\forall uy[\neg R(x,u) \vee \neg R(u,y) \vee Q_{\neg\Box r,p}(y)] \vee \forall z[\neg R(x,z) \vee Q_{\Box r,\neg p}(z)])$ $\forall x(\forall uy[\neg R(x,u) \vee \neg R(u,y) \vee \neg Q_{\neg\Box r,p}(y)] \vee \forall vz[\neg R(x,v) \vee \neg R(v,z) \vee Q_{\Box r,\neg p}(z)])$	$\Box r,\neg p$

Notice

- **restrictions on accessibility relation R**
- **new predicate terms arise like**

  $\mathbf{\neg Q_{\Box p}}$ **and** $\mathbf{Q_{\Box p}}$

Derivation of Axiomatic Translations of Common Axioms:

$\forall p\forall x(\pi(\neg p,x) \leftrightarrow \neg\pi(p,x))$ [1]

$\forall pq\forall x(\pi(p*q,x) \leftrightarrow (\pi(p,x)*\pi(q,x)))$ [2] where $* \in \{\rightarrow,\leftrightarrow,\vee,\wedge\}$

$\forall p\forall x(\pi(\Box p,x) \leftrightarrow \forall y(R(x,y) \rightarrow \pi(p,y)))$ [3]

Axiom 5: $\neg\Box\neg\Box p \rightarrow \Box p$

$\forall p\forall x(\pi(\neg\Box\neg\Box p \rightarrow \Box p,x) \equiv \forall p\forall x(\pi(\neg\Box\neg\Box p,x) \rightarrow \pi(\Box p,x))$ [from 2]

$\equiv \forall p\forall x(\neg\pi(\Box\neg\Box p,x) \rightarrow \pi(\Box p,x))$ [from 1]

$\equiv \forall p\forall x(\neg\forall y(R(x,y) \rightarrow \pi(\neg\Box p,y)) \rightarrow \pi(\Box p,x))$ [from 3]

$\equiv \forall p\forall x(\neg\forall y(R(x,y) \rightarrow \neg\pi(\Box p,y)) \rightarrow \pi(\Box p,x))$ [from 1]

Re-writing gives $\forall x(\neg\forall y(R(x,y) \rightarrow \neg Q_{\Box p}(y)) \rightarrow Q_{\Box p}(x))$

$\equiv \forall x(\neg\neg\forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y)) \vee Q_{\Box p}(x))$

$\equiv \mathbf{\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y)) \vee Q_{\Box p}(x))}$

Axiomatic Translation of Modal Axioms (2)

- For each modal-axiom a new term is added ($\wedge$) to output
- Use edited version the instantiation set, in which only sub-terms immediately below a box ($\Box$) symbol are included …. $\Box\psi \in Sf(\varphi)$
- Each sub-term of the edited-instantiation set is translated by applying the stated formula e.g. Axiom 5: $\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box r,p}(y)) \vee Q_{\Box r,p}(x))$

Example $\varphi = \Box((\neg\Box(r,p)) \wedge \Box(r,q))$ in KB [recall B is $\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box r,p}(y)) \vee Q_{r,p}(x))$]

The edited instantiation set is

1. $\Box((\neg\Box p)\wedge\Box q)$ 2. $\Box p$ 3. $\Box q$

So φ in KB is

	$\prod(\Box((\neg\Box p)\wedge\Box q))$	[as defined above]
$\wedge$	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box(\neg\Box p\wedge\Box q)}(y)) \vee Q_{\neg\Box p\wedge\Box q}(x))$	**B($\Box(\neg\Box p\wedge\Box q)$)**
$\wedge$	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y)) \vee Q_{p}(x))$	**B($\Box p$)**
$\wedge$	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box q}(y)) \vee Q_{q}(x))$	**B($\Box q$)**

- New predicate symbols may need to be *Defined* – that is translated in Def(ψ)
- For axiom D, the translation is $\neg Q_{\Box\phi}(x) \vee Q_{\neg\Box\neg\phi}(x)$
- The predicate $Q_{\neg\Box\neg\phi}$ is new, so the new term $\Box\neg\phi$ that needs to be Defined

φ in KD is

	$\prod(\Box((\neg\Box p)\wedge\Box q))$	[as defined above]
$\wedge$	$\forall x(\neg Q_{\Box(\neg\Box p\wedge\Box q)}(x) \vee Q_{\neg\Box\neg(\neg\Box p\wedge\Box q)}(x))$	[D($\Box((\neg\Box p)\wedge\Box q)$)]
$\wedge$	$\forall x(\neg Q_{\Box p}(x) \vee Q_{\neg\Box\neg p}(x))$	[D($\Box p$)]
$\wedge$	$\forall x(\neg Q_{\Box q}(x) \vee Q_{\neg\Box\neg q}(x))$	[D($\Box q$)]
$\wedge$	$\forall x(Q_{\Box\neg(\neg\Box p\wedge\Box q)}(x) \rightarrow \forall y(R(x,y) \rightarrow Q_{\neg(\neg\Box p\wedge\Box q)}(y)))$	**[Def($\Box\neg(\neg\Box p\wedge\Box q)$)]**
$\wedge$	$\forall x(Q_{\Box\neg(\neg\Box p\wedge\Box q)}(x) \leftrightarrow \neg Q_{\neg\Box\neg(\neg\Box p\wedge\Box q)}(x))$	
$\wedge$	$\forall x(Q_{\neg\Box\neg(\neg\Box p\wedge\Box q)}(x) \rightarrow \exists y(R(x,y) \wedge Q_{\neg\Box p\wedge\Box q}(y)))$	
$\wedge$	$\forall x(Q_{\Box\neg p}(x) \rightarrow \forall y(R(x,y) \rightarrow Q_{\neg p}(y)))$	**[Def($\Box\neg p$)]**
$\wedge$	$\forall x(Q_{\Box\neg p}(x) \leftrightarrow \neg Q_{\neg\Box\neg p}(x))$	
$\wedge$	$\forall x(Q_{\neg\Box\neg p}(x) \rightarrow \exists y(R(x,y) \wedge Q_{p}(y)))$	
$\wedge$	$\forall x(Q_{\Box\neg q}(x) \rightarrow \forall y(R(x,y) \rightarrow Q_{\neg q}(y)))$	**[Def($\Box\neg q$)]**
$\wedge$	$\forall x(Q_{\Box\neg q}(x) \leftrightarrow \neg Q_{\neg\Box\neg q}(x))$	
$\wedge$	$\forall x(Q_{\neg\Box\neg q}(x) \rightarrow \exists y(R(x,y) \wedge Q_{q}(y)))$	

Compositional Terms for modal axioms 5 and 5^K

For axiom 5

- The instantiation set ($\chi^{\in}_{\varphi}$) of sub-formula below a box ($\Box$) symbol is <u>not</u> complete
- Additional term of form $\Box\neg\Box\psi$ for each sub-formula $\Box\psi$ must be included in the instantiation set, and *Defined*, so : $\Box\chi^{5}_{\varphi} = \Box\chi^{\in}_{\varphi} \cup \Box\neg\Box\chi^{\in}_{\varphi}$
- The default translation of axiom 5 is $K5_o$ indicating inclusion of compositional terms

Example $\neg\Box\Box\Box p \wedge \Box\Box p$* (in K5 satisfiable; but in $K5_o$ unsatisfiable).

$\Box\chi^{5}_{\varphi}$ is $\{\Box\Box\Box p, \Box\Box p, \Box p\}$, and $Sf(\varphi) = \{\neg\Box\Box\Box p \wedge \Box\Box p, \Box\Box\Box p, \Box\Box p, \Box p, p\}$

and the translation is $\prod(\neg\Box\Box\Box p \wedge \Box\Box p)$

$\wedge\ 5(\Box\Box\Box p) \wedge 5(\Box\Box p) \wedge 5(\Box p)$

$\wedge$ **$Def(\Box\neg\Box\Box\Box p) \wedge Def(\Box\neg\Box\Box p) \wedge Def(\Box\neg\Box p)$**

$\wedge$ **$5(\Box\neg\Box\Box\Box p) \wedge 5(\Box\neg\Box\Box p) \wedge 5(\Box\neg\Box p)$**

Similar concerns apply to axiom 5^{κ}

$\neg\Box\Box\Box p \wedge \Box\Box p$ in $K5^3$ the translation is

$\prod(\neg\Box\Box\Box p \wedge \Box\Box p)$

$\wedge\ 5^3(\Box\Box\Box p) \wedge 5^3(\Box\Box p) \wedge 5^3(\Box p)$

$\wedge\ Def(\Box\Box\Box\neg\Box\Box\Box p) \wedge Def(\Box\Box\Box\neg\Box\Box p) \wedge Def(\Box\Box\Box\neg\Box p)$

$\wedge$ **$Def(\Box\Box\neg\Box\Box\Box p)^* \wedge Def(\Box\Box\neg\Box\Box p)^* \wedge Def(\Box\Box\neg\Box p)^*$**

$\wedge$ **$Def(\Box\neg\Box\Box\Box p)^* \wedge Def(\Box\neg\Box\Box p)^* \wedge Def(\Box\neg\Box p)^*$**

$\wedge\ 5^3(\Box\Box\Box\neg\Box\Box\Box p) \wedge 5^3(\Box\Box\Box\neg\Box\Box p) \wedge 5^3(\Box\Box\Box\neg\Box p)$

$\wedge$ **$5^3(\Box\Box\neg\Box\Box\Box p)^* \wedge 5^3(\Box\Box\neg\Box\Box p)^* \wedge 5^3(\Box\Box\neg\Box p)^*$**

$\wedge$ **$5^3(\Box\neg\Box\Box\Box p)^* \wedge 5^3(\Box\neg\Box\Box p)^* \wedge 5^3(\Box\neg\Box p)^*$**

Notice the great increase in the number of terms

Composition in a Series of Modal Axioms

Composition of the Axiomatic Translation of Common Modal Axioms

Axiom	Modal representation of axiom	For composition
T_o	$\Box p \rightarrow p$	Nothing
B_o	$\neg\Box\neg\Box p \rightarrow p$	$\Box\neg\Box p$
D_o	$\Box p \rightarrow \neg\Box\neg p$	$\Box\neg p$
4^K_o	$\Box p \rightarrow \Box k\Box p$	$\Box^K\Box p$
4_o	$\Box p \rightarrow \Box\Box p$	$\Box\Box p$
4^2_o	$\Box p \rightarrow \Box\Box\Box p$	$\Box\Box\Box p$
4^3_o	$\Box p \rightarrow \Box\Box\Box\Box p$	$\Box\Box\Box\Box p$
5^K_o	$\neg\Box k\neg\Box p \rightarrow \Box p$	$\Box^K\neg\Box p$
5_o	$\neg\Box\neg\Box p \rightarrow \Box p$	$\Box\neg\Box p$
5^2_o	$\neg\Box\Box\neg\Box p \rightarrow \Box p$	$\Box\Box\neg\Box p$
5^3_o	$\neg\Box\Box\Box\neg\Box p \rightarrow \Box p$	$\Box\Box\Box\neg\Box p$
$alt_1^{K1K2}{}_o$	$\neg\Box^{K1}\Box p \rightarrow \Box^{K2}\Box\neg p$	$\Box^{K2}\Box\neg p$ & $\Box^{K1}\Box p$
alt_{1o}	$\neg\Box p \rightarrow \Box\neg p$	$\Box\neg p$
$alt_1^{1,1}{}_o$	$\neg\Box\Box p \rightarrow \Box\Box\neg p$	$\Box\Box p$ & $\Box\Box\neg p$
$alt_1^{1,2}{}_o$	$\neg\Box\Box p \rightarrow \Box\Box\Box\neg p$	$\Box\Box p$ & $\Box\Box\Box\neg p$
$alt_1^{2,1}{}_o$	$\neg\Box\Box\Box p \rightarrow \Box\Box\neg p$	$\Box\Box\Box p$ & $\Box\Box\neg p$
$alt_1^{2,2}{}_o$	$\neg\Box\Box\Box p \rightarrow \Box\Box\Box\neg p$	$\Box\Box\Box p$ & $\Box\Box\Box\neg p$

Composition Mechanism

- To ensure **completeness,** axiom combination $K4_oB$, requires instantiation sets $\chi_4 = \chi^{\in}_{\varphi}$ & $\chi_B = \chi^4_{\varphi}$
- Example $\varphi = \neg(\Box\neg\Box p \vee \Box p)$ in K4B
  - $Sf(\varphi) = \{ \neg\Box\neg\Box p \wedge \neg\Box p, \Box\neg\Box p, \Box p, p \}$

    $\Box\chi = \{ \Box\neg\Box p, \Box p \}$
  - Translation

    $\Pi(\Box\neg\Box p \vee \Box p)$

    $\wedge\ 4(\Box\neg\Box p) \wedge 4(\Box p) \wedge Def(\Box\Box\neg\Box p) \wedge Def(\Box\Box p)$**

    $\wedge\ B(\Box\neg\Box p) \wedge B(\Box p) \wedge$ **$B(\Box\Box\neg\Box p) \wedge B(\Box\Box p)$**
  - At ** instantiation set is updated to

    $\Box\chi = \{\Box\neg\Box p, \Box p, \Box\Box\neg\Box p, \Box\Box p\}$
  - How is this derived? $\varphi = \Box\neg\Box p \vee \Box p$
- Similar considerations apply to the combination $KT4_oB$
- **with composition order** of axioms in a sequence is important since instantiation set updated *in light of* previous axioms
- all axiom combinations *could* use composition
  - But $KD_oB = KDB$ and $KD_o4 = KD4$
  - Inclusion of unnecessary terms extends problem, and usually lengthen proof

extended-SPASS Syntax

- extended-SPASS offers a mode in which axiom combinations can be defined for which result has been proven complete
- in experimental mode, *any* axiom combination can be defined but this is not guaranteed complete
- 26 modal axioms implemented (so far)

```
begin_problem(zzzzzz).
list_of_descriptions.
  name({* Problem:KK-bp,ddnp,g *}).
  author({* *}).
  status(unknown).
  description({* *}).
end_of_list.

list_of_symbols.
predicates[(R,2),(r,0),(p,0),(q1,0),(q2,0)].
end_of_list.

list_of_special_formulae(axioms,eml).
prop_formula(
and(and(box(r,p),dia(r,dia(r,not(p)))),and(dia(r,dia(r,dia(r,dia(r,dia(
r,dia(r,q1))))))),dia(r,dia(r,dia(r,dia(r,dia(r,dia(r,q2)))))))
).
end_of_list.

list_of_special_formulae(conjectures,EML).
end_of_list.

list_of_settings(SPASS).
{*
set_flag(EMLAxiom,1).
set_flag(DocProof,0).
set_flag(Sorts,0).
set_flag(CNFRenaming,0).
set_flag(CNFOptSkolem,0).
set_flag(CNFStrSkolem,0).
set_flag(Select,0).
set_flag(Splits,-1).
set_flag(PGiven,0).
set_flag(PProblem,0).
*}
end_of_list.
```

standard options for SPASS

- *Axiom* $K5_oB$ *is defined by the flags combination*

  `'-EMLAxiom=1 –EMLAxiomLog5=1 –EMLAxiomLogB=1'`

- *Reported as*

  `Requested Composition ..... K,5o,B`

- *Axiom combination* $K.4c.5o.Alt_1^{2,1}$ *is accessed using the flags combination*

  `-EMLAxiom=1 –EMLAxiomLog5=2 –EMLClassLog4=1`
  `–EMLAxiomLogAlt1KK=3 EMLAxiomLogFac1=2 EMLAxiomLogFac2=1`
  `–EMLAxiomHidOpt=901000000`

- *Reported to the user*

  `Building a General Composition Structure`
  `Requested Composition ..... K,4c,5o,ALT1KKo[2][1]`

```
list_of_settings(SPASS).
{*
set_flag(EMLAxiom,1).
set_axiomflag([r,Bo],[r,5o],[r,T]).
*}
```

Multiple Axioms and Conjectures in input problem are automatically reorganized

$(\mathbf{A}_1 \wedge \mathbf{A}_2 \wedge \ldots \mathbf{A}_n) \wedge (\neg\mathbf{C}_1 \wedge \neg\mathbf{C}_2 \wedge \ldots \neg\mathbf{C}_n))$

Typical .dfg template input file for testing axiomatic translation, for the problem bp,ddnp,g.dfg

$(\wedge(\wedge(\Box(r,p),\Diamond(r,\Diamond(r,\neg p))),\wedge(\Diamond(r,\Diamond(r,\Diamond(r,\Diamond(r,\Diamond(r,\Diamond(r,q_1))))))),\Diamond(r,\Diamond(r,\Diamond(r,\Diamond(r,\Diamond(r,\Diamond(r,\Diamond(r,q_2)))))))))).$

Results and Execution times for Axiomatic Translation of 119 problems

200 seconds execution time; Completion (Satisfiable); Proof (Unsatisfiable)

Axiom	Total No. Problems Solved (failed)	Execution Time in msec Median	Mean±SD	Max	Proof No. Problems Solved	Execution Time in msec Median	Mean±SD	Max	Completion No. Problems Solved	Execution Time in msec Median	Mean±SD	Max
K	119(0)	20	30±33	170	4	15	53±78	170	115	20	29±30	170
T	119(0)	50	61±37	200	15	50	49±20	90	104	60	62±39	200
B	119(0)	20	93±330	3120	21	20	34±26	100	98	20	106±363	3120
D	119(0)	30	103±188	1050	8	10	56±119	350	111	40	106±192	1050
4	119(0)	20	305±1235	8750	26	20	24±20	100	93	20	383±1388	8750
5	119(0)	60	665±2429	19320	53	40	113±162	670	66	110	1109±3200	19320
alt_1	119(0)	30	124±235	1310	19	30	49±94	430	100	40	138±251	1310
4^2	117(2)	40	2425±10127	72670	13	20	48±80	310	104	50	2722±10710	72670
4^3	90(29)	155	8908±26582	135460	10	35	777±2334	7420	80	210	9924±28036	135460
5^2	73(46)	620	13544±27403	120930	27	190	2742±10856	56810	46	4165	19884±31973	120930
5^3	23(96)	1450	14500±40754	194120	17	2450	18362±46974	194120	6	745	3555±7229	18240
$alt_1^{1,1}$	87(32)	180	13855±32783	182750	12	45	72±72	240	75	380	16061±34832	182750
$alt_1^{2,1}$	85(34)	340	15178±38295	197220	30	140	3949±18837	103520	55	650	21303±44524	197220
$alt_1^{1,2}$	66(53)	440	6383±16898	108990	24	450	7123±23517	108990	42	440	5961±11915	47520
$alt_1^{2,2}$	26(93)	5835	11577±19485	75660	8	2775	19574±31956	75660	18	8170	8023±9715	34990
T4	119(0)	20	72±175	1250	42	20	29±23	120	77	30	96±213	1250
TB	119(0)	20	67±198	1820	30	20	30±28	120	89	20	80±227	1820
DB	119(0)	40	872±5061	50190	25	30	107±168	680	94	40	1075±5682	50190
D4	119(0)	40	483±1823	14530	33	30	53±84	430	86	50	648±2124	14530
4B	119(0)	30	340±1423	13020	87	30	61±89	540	32	65	1100±2622	13020
5_oB	115(4)	90	717±2358	16750	86	80	515±1968	15680	29	160	1319±3222	16750
T5	119(0)	100	1385±9280	96470	93	80	180±234	1190	26	150	5572±19562	96470
T4B	119(0)	40	429±2158	20940	93	30	53±68	500	26	90	1774±4423	20940
D4B	119(0)	60	1798±9000	78010	93	50	133±235	1220	26	180	7752±18300	78010

- **Problems with axioms 4, 5, alt_1 are solved** **("reduced to decidable FOL fragments")**
- **only a partial solution to the problem where chains of predicates are still present in the translation (4^K, 5^K, alt_1^{KK})**

Number of test examples (from 119) for classical & axiomatic translations with the (i) same (ii) greater (iii) lower execution times, and the ratios of execution times

C = Completion (Satisfiable); P = Proof (Unsatisfiable)

time Classical = time Axoimatic	Proof	Complete	+	time Classical > time Axiomatic	Proof	Complete	+	mean ratios(P&C)	failed only in classical Proof	Complete	time Axiomatic > time Classical	Proof	Complete	+
T		11	11	T	2	5	7	1.9 & 2	0	0	T	13	88	101
B	11	46	57	B	10	49	59	1.5 & 1	0	0	B		3	3
D	4	18	22	D	1	1	2	2 & 3	0	0	D	3	92	95
4	6		6	4	19		19	21.9	1	93	4			0
5	13		13	5	27		27	7.4	5	66	5	8		8
alt1	4		4	alt1	13		13	2.2	1	100	alt1	1		1
4^2	2		2	4^2	9		9	620.7	2	104	4^2			0
4^3	2		2	4^3	3		3	1205.3	5	80	4^3			0
5^2			0	5^2	11		11	215	13	46	5^2	3		3
5^3			0	5^3			0		14	6	5^3	3		3
alt1^1,1	2		2	alt1^1,1	6		6	807.2	4	75	alt1^1,1			0
alt1^2,1	2		2	alt1^2,1	3		3	24.5	25	55	alt1^2,1			0
alt1^1,2	2		2	alt1^1,2	1		1	3.3	21	42	alt1^1,2			0
alt1^2,2	2		2	alt1^2,2	1		1	2.3	5	18	alt1^2,2			0
T4	10		10	T4	22		22	763.9	7	77	T4	3		3
TB	19	36	55	TB	11	46	57	1.8 & 2	0	0	TB		7	7
DB	10	27	37	DB		4	4	1	0	0	DB	15	63	78
D4	5		5	D4	22		22	433.9	5	72	D4	1	14	15
4B	13		13	4B	46		46	24.4	18	32	4B	10		10
5oB	9		9	5oB	35		35	39.2	15	29	5oB	27		27
T5	9		9	T5	39		39	13.1	16	26	T5	29		29
T4B	17		17	T4B	46		46	81.9	25	19	T4B	5	7	12

- **Improvements for axiomatic translation for triangular axioms and for satisfiable problems**

- 119+ template .dfg files formulated, and executed in extended-SPASS with several dozen single axioms & axiom combinations applied in turn using flag-based syntax

Syntax / usage note

```
#!/bin/csh
setenv t 200
cd template
foreach f ( *.dfg )
        nice SPASS -TimeLimit=$t -EMLAxiomLogT=1     $f > ../out/`basename $f .dfg`.T.out
        nice SPASS -TimeLimit=$t -EMLAxiomLogD=1     $f > ../out/`basename $f .dfg`.D.out
        nice SPASS -TimeLimit=$t                     $f > ../out/`basename $f .dfg`.K.out
...etc...
end
```

Examples of usage

Summary of Counter-Examples (formulations that are *not* complete)

The formulae that show discriminaition between complete and non-complete formulations of the axiom combination are shown. Non-complete examples are in brackets.

Discrimination between formulation of modal axiom	Formula
[5] vs 5_o	bbp,dddd(q,dnp); dddddnq,dp,bbp; dddddnq,bbbq; ddddnq,bbp; ddddnq,dp,bbp; dddnq,bbq; dddnq,dp,bbp
[5^2] vs 5^2_o	dddddnq,dp,bbp; dddddnq,bbbq; ddddnq,bbp; ddddnq,dp,bbp
[5^3] vs 5^3_o	
[4B / B_o4] vs 4_oB	5; aiml02_prop3iii; amai02b; dnq,dbq
[45B]	
[5B / B_o5_o] vs 5_oB	4; 4^2; 4^3; amai02; bbp,dddd(q,dnp); bp,dddnp,g; bp,ddnp,g; bq,dddddddnq; bq,dddddnq; dddddnq,dp,bbp; dddddnq,bbbq; ddddnq,b(q,g); ddddnq,bbp; ddddnq,bq; ddddnq,dp,bbp; dddnq,bbq; dddnq,bq; dddnq,dp,bbp; ddnq,bq; demri1; demri2
[T4B / TBo4 / Bo4oT / BoT4] vs T4oB / 4oBoT / 4oTB	5; aiml02_prop3iii; amai02b; dnq,dbq
T vs DT	None
[$T5_o$] vs 5_oT	4; 4^2; 4^3; amai02; bp,dddnp,g; bq,dddddddnq; bq,dddddnq; ddddnq,b(q,g); ddddnq,bq; dddnq,bq; ddnq,bq; demri1; demri2
[TB5 / TB_o5 / B_o5_oT] vs $T5_oB$ / 5_oB_oT / 5_oTB / B_oT5	4; 4^2; 4^3; amai02; bbp,dddd(q,dnp); bp,dddnp,g; bp,ddnp,g; bq,dddddddnq; bq,dddddnq; dddddnq,dp,bbp; dddddnq,bbbq; ddddnq,b(q,g); ddddnq,bbp; ddddnq,bq; ddddnq,dp,bbp; dddnq,bbq; dddnq,bq; dddnq,dp,bbp; ddnq,bq; demri1; demri2
[D5B / B_o5_oD / B_oD_o5 / D_oB_o5] vs D_o5_oB / 5_oD_oB / 5_oB_oD	4; 4^2; 4^3; amai02; bq,dddddddnq; bq,dddddnq; ddddnq,b(q,g); ddddnq,bq; dddnq,bq; ddnq,bq; demri1 partial… bbp,dddd(q,dnp); bp,dddnp,g; bp,ddnp,g; dddddnq,dp,bbp; dddddnq,bbbq; ddddnq,bbp; ddddnq,dp,bbp; dddnq,bbq; dddnq,dp,bbp; demri2
[D4B / B_oD_o4 / B_o4_oD] vs D_oB_o4/ D_o4_oB / 4_oD_oB / 4_oB_oD	5; aiml02_prop3iii; amai02b; dnq,dbq

Screening for Complete Varieties of S5:

$T4_oB$ 4_oTB 4_oB_oT
D_o4_oB D_oB_o4 4_oD_oB 4_oB_oD
5_oT
$T5_oB$ 5_oB_oT 5_oTB B_oT5
4_oTB_o5 4_oT5_oB $4_o5_oB_oT$ 4_o5_oTB 4_oB_oT5 $4_oB_o5_oT$
B_o4_oT5 $B_o4_o5_oT$ B_o5_oT4 $B_o5_o4_oT$ B_oT4_o5 B_oT5_o4
$5_o4_oB_oT$ 5_o4_oTB 5_oTB_o4 5_oT4_oB $5_oB_o4_oT$ 5_oB_oT4
T4B5 $T4_oB_o5$ $T4_o5_oB$ $T5_oB_o4$ $T5_o4_oB$ TB_o4_o5 TB_o5_o4
T45 $T4_o5$ $T5_o4$ 4_oT5 4_o5_oT 5_oT4 5_o4_oT
D4B5 $D_o4_oB_o5$ $D_o4_o5_oB$ $D_o5_oB_o4$ $D_o5_o4_oB$ $D_oB_o4_o5$ $D_oB_o5_o4$
$4_oD_oB_o5$ $4_oD_o5_oB$ $4_o5_oB_oD$ $4_o5_oD_oB$ $4_oB_oD_o5$ $4_oB_o5_oD$
$B_o4_oD_o5$ $B_o4_o5_oD$ $B_o5_oD_o4$ $B_o5_o4_oD$ $B_oD_o4_o5$ $B_oD_o5_o4$
$5_o4_oB_oD$ $5_o4_oD_oB$ $5_oD_oB_o4$ $5_oD_o4_oB$ $5_oB_o4_oD$ $5_oB_oD_o4$
D_o5_oB 5_oD_oB 5_oB_oD

Which formulation of S5 is solved fastest?

Axiom	mean execution (msec)
T54	7550
T54B	19410
T4oB	77130
4oTB	82120
5oT	95370
D54B	127030
Bo5oT4	144120
5oT4	187850

Multi-modal Formulae

- Distinguish between different instances of the same modal operator
- A modality index is attached to each modal operator $\Box(r,p)$ and $\Diamond(r,p)$
- Indicates its scope and allegiances within the modal formula
- Each modality is associated with a unique accessibility relationship
- For example $\Box(\mathbf{r}, (\neg\Box(\mathbf{s},p) \wedge \Box(\mathbf{t},q)))$
- Sub-formulae $\{ \Box(r, (\neg\Box(s,p) \wedge \Box(t,q))), \neg\Box(s,p) \wedge \Box(t,q), \Box(s,p), \Box(t,q), p, q \}$
- Use **R**, **S**, and **T** for accessibility relations corresponding to modality indices r, s, t

$\Pi(\Box(r, (\neg\Box(s,p) \wedge \Box(t,q))))$ = $\exists x Q_{\Box(r,(\neg\Box(s,p)\wedge\Box(t,q)))}(x)$

$\wedge$ $\forall x(Q_{\Box(r,(\neg\Box(s,p)\wedge\Box(t,q)))}(x) \rightarrow \forall y(R(x,y) \rightarrow Q_{\neg\Box(s,p)\wedge\Box(t,q)}(y))$ [Def($\Box(r,(\neg\Box(s,p)\wedge\Box(t,q))))$]

$\wedge$ $\forall x(Q_{\Box(r,(\neg\Box(s,p)\wedge\Box(t,q)))}(x) \leftrightarrow \neg Q_{\neg\Box(\neg\Box p\wedge\Box q)}(x))$

$\wedge$ $\forall x(Q_{\neg\Box(r,(\neg\Box(s,p)\wedge\Box(t,q)))}(x) \rightarrow \exists y(R(x,y) \wedge Q_{\neg(\neg\Box(s,p)\wedge\Box(t,q))}(y))$

$\wedge$ $\forall x(Q_{\neg\Box(s,p)\wedge\Box(t,q)}(x) \rightarrow (Q_{\neg\Box(s,p)}(x) \wedge Q_{\Box(t,q)}(x))$ [Def($\neg\Box(s,p)\wedge\Box(t,q)$)]

$\wedge$ $\forall x(Q_{\neg\Box(s,p)\wedge\Box(t,q)}(x) \leftrightarrow \neg Q_{\neg(\neg\Box(s,p)\wedge\Box(t,q))}(x))$

$\wedge$ $\forall x(Q_{\neg(\neg\Box(s,p)\wedge\Box(t,q))}(x) \rightarrow (Q_{\Box(s,p)}(x) \vee Q_{\neg\Box(t,q)}(x))$

$\wedge$ $\forall x(Q_{\Box(s,p)}(x) \rightarrow \forall y(S(x,y) \rightarrow Q_p(y))$ [Def($\Box(s,p)$)]

$\wedge$ $\forall x(Q_{\Box(s,p)}(x) \leftrightarrow \neg Q_{\neg\Box(s,p)}(x))$

$\wedge$ $\forall x(Q_{\neg\Box(s,p)}(x) \rightarrow \exists y(S(x,y) \wedge Q_{\neg p}(y))$

$\wedge$ $\forall x(Q_{\Box(t,q)}(x) \rightarrow \forall y(T(x,y) \rightarrow Q_q(y))$ [Def($\Box(t,q)$)]

$\wedge$ $\forall x(Q_{\Box(t,q)}(x) \leftrightarrow \neg Q_{\neg\Box(t,q)}(x))$

$\wedge$ $\forall x(Q_{\neg\Box(t,q)}(x) \rightarrow \exists y(T(x,y) \wedge Q_{\neg q}(y))$

$\wedge$ $\forall x(Q_p(x) \leftrightarrow \neg Q_{\neg p}(x)) \wedge \forall x(Q_q(x) \leftrightarrow \neg Q_{\neg q}(x))$ [from Def(p) & Def(q)]

extended-SPASS syntax

```
list_of_settings(SPASS).
{*
set_flag(EMLAxiom,1).
set_axiomflag([r,Bo],[s,5o],[r,T],[s,T]).
*}
```

- Particularly important when it comes to applying axioms
- Instantiation set for each axiom *edited* according to modality index
- When axioms applied to different modalities, they do *not* interact via instantiation set

For example $\Pi(\Box(r, (\neg\Box(s,p) \wedge \Box(t,q))))$

$\wedge$ $\forall x(\neg Q_{\Box(r,(\neg\Box(s,p)\wedge\Box(t,q)))}(x) \vee Q_{\neg\Box(s,p)\wedge\Box(t,q)}(x))$ T,r($\Box(r, (\neg\Box(s,p) \wedge \Box(t,q))))$

$\wedge$ $\forall x(\forall y(\neg S(x,y) \vee \neg Q_{\Box(s,p)}(y)) \vee Q_p(x))$ B,s($\Box(s,p)$])

$\wedge$ $\forall x(\neg Q_{\Box(t,q)}(x) \vee Q_{\neg\Box(t,\neg q)}(x))$ D,t($\Box(t,q)$])

$\wedge$ Def($\Box(t,\neg q)$)

Bi-modal Axioms

Axiom	Correspondence Property	AxiomaticTranslation		
		Translation	New Term	Composition Term
CR $[r]p \rightarrow [s]p$		$\forall x(\neg Q_{[r]p}(x) \vee Q_{[s]p}(x))$	None	None
CR2 $[r]p \rightarrow [s][r]p$	$(R_s(x,y) \wedge R_r(y,z)) \rightarrow R_r(x,z)$	$\forall x(\neg Q_{[r]p}(x) \vee \forall y(\neg R_s(x,y) \vee Q_{[r]p}(y)))$	None	$[s][r]p$
CR3 $[r]p \rightarrow [r][s]p$	$(R_r(x,y) \wedge R_s(y,z)) \rightarrow R_r(x,z)$	$\forall x(\neg Q_{[r]p}(x) \vee \forall y(\neg R_r(x,y) \vee Q_{[s]p}(y)))$	None	$[r][s]p$

extended-SPASS syntax

```
set_axiom([r,s,Y],[r,X]).
```

- Bimodal axiom Y, where r and s are different modality indices

---

Not Covered here

- Fine control over translation - many different options available
- Information about the translation displayed - to help novice users
- Details of the optimizations used
- Experimental Implementation of an Axiomatic Translation of a General Modal Axiom (currently in preparation)
- Mixed axioms (e.g. T_c4B_c, D_cB, D_c4)

Not Considered

- n-ary axioms (referring to more than one modal formula)

e.g. Hintikka $\neg\Box(\Box p \rightarrow q) \rightarrow \Box(\Box q \rightarrow p)$; $(R(x,y) \wedge R(x,z)) \rightarrow (R(y,z) \vee R(z,y))$; $\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y) \vee Q_q(y)) \vee \forall y(\neg R(x,y) \vee \neg Q_{\Box q}(y) \vee Q_p(y))$

Summary of the Classical Translations (Correspondence Properties) and Axiomatic Translations of Some More Modal Axioms

Axiom	Correspondence Property***	AxiomaticTranslation		
		Translation	New Term	Composition Term
Shift Reflexive (SR) $\Box(\Box p \rightarrow p)$	$R(x,y) \rightarrow R(y,y)$	$\forall x \forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y) \vee Q_p(y))$	None	$\Box(\Box p \rightarrow p)$
Convergence (G)* $\Diamond\Box p \rightarrow \Box\Diamond p$	$R(y,z) \wedge R(y,u) \rightarrow \exists x(R(z,x) \wedge R(u,x))$	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y)) \vee \forall y(\neg R(x,y) \vee \neg Q_{\Box \neg p}(y))$	$\Box\neg p$	$\Box\neg\Box p$, $\Box\neg\Box\neg p$
Dense (Den)** $\Box\Box p \rightarrow \Box p$	$R(y,z) \rightarrow \exists x(R(x,z) \wedge R(y,x))$	$\forall x(\exists y(R(x,y) \wedge \neg Q_{\Box p}(y)) \vee Q_{\Box p}(x))$	None	$\Box\Box p$
Trivial (Tr) $\Box p \leftrightarrow p$	$R(x,y) \leftrightarrow x \approx y$	$\forall x((\neg Q_{\Box p}(x) \vee Q_p(x)) \wedge (\neg Q_p(x) \vee Q_{\Box p}(x)))$	None	None
McKinsey (M) $\Box\Diamond p \rightarrow \Diamond\Box p$	-	$\forall x(\exists y(R(x,y) \wedge Q_{\Box \neg p}(y)) \vee \exists y(R(x,y) \wedge Q_{\Box p}(y)))$	$\Box\neg p$	$\Box\neg\Box p$, $\Box\neg\Box\neg p$
Löb (W) $\Box(\Box p \rightarrow p) \rightarrow \Box p$	-	$\forall x(\exists y(R(x,y) \wedge Q_{\Box p}(y) \wedge \neg Q_p(y)) \vee Q_{\Box p}(x))$	None	$\Box(\Box p \rightarrow p)$
B^2 $\Diamond\Diamond\Box p \rightarrow p$	$(R(x,y) \wedge R(y,z)) \rightarrow R(z,x)$	$\forall x(\forall y(\neg R(x,y) \vee \forall z(\neg R(y,z) \vee \neg Q_{\Box p}(z))) \vee Q_p(x))$	None	$\Box\Box\neg\Box p$, $\Box\neg\Box p$
B^3 $\Diamond\Diamond\Diamond\Box p \rightarrow p$		$\forall x(\forall y(\neg R(x,y) \vee \forall z(\neg R(y,z) \vee \forall u(\neg R(z,u) \vee \neg Q_{\Box p}(u)))) \vee Q_p(x))$	None	$\Box\Box\Box\neg\Box p$, $\Box\Box\neg\Box p$, $\Box\neg\Box p$
DBBB $\Diamond\Box p \rightarrow \Box\Box p$	$(R(x,y) \wedge R(x,u) \wedge R(u,z)) \rightarrow R(y,z)$	$\forall x(\forall y(\neg R(x,y) \vee \neg Q_{\Box p}(y)) \vee \forall y(\neg R(x,y) \vee Q_{\Box p}(y)))$	None	$\Box\neg\Box p$, $\Box\Box p$

*In Scott-Lemmon with h=i=j=k=1

** In Scott-Lemmon with h=k=0, i=2, j=1

***If not otherwise stated all variables are universally quantified

- **A new modal axiom takes less than *1 hour* to implement / test**
- **Testing is simpler if a correspondence property can be defined**
- **Implemented in a similar fashion to axiom 5_o with 'integral composition', and with shortcuts** (no proof of more restricted instantiation set needed)
- Automated version in preparation (likely to be non-optimal translations)

Examples of the results of testing for new axioms

more at http://project.kjsmith.net

Results for the test formulae in each of the listed axioms (Continued).

C is Completion in SPASS (satisfiable), P is Proof in SPASS (unsatisfiable), and X is ran out of time at 200 seconds execution time.

		B^2	B^2_o	B^2_c	B^3	B^3_o	B^3_c	DBBB	$DBBB_o$	$DBBB_c$	DEN	DENo	DENc	DDEN	DoDENo	DEND	DoENDo	G	Go	Gc	SR	SRo	SRc	TR	TR_o	TR_c
1	4	C	C	X	C	X	X	C	C	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
2	4^2	C	C	X	C	X	X	C	C	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
3	4^3	C	P	X	C	X	X	C	C	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
4	5	C	C	X	C	X	X	C	C	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
5	5^2	C	C	X	C	X	X	C	C	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
6	5^3	C	P	P	C	X	X	C	C	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
7	D	C	C	X	C	X	X	C	C	X	C	C	C	P	P	P	P	C	C	X	C	C	C	P	P	P
8	T	C	C	X	C	C	X	C	C	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
9	alt1	C	C	X	C	X	X	C	C	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
10	alt1^1,1	C	X	X	C	X	X	C	C	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
11	alt1^1,2	C	X	X	C	X	X	C	X	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
12	alt1^2,1	C	X	X	C	X	X	C	X	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
13	alt1^2,2	C	X	X	C	X	X	C	X	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
14	B	C	C	X	C	X	X	C	C	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
15	B^2	P	P	P	C	X	X	C	C	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
16	F	C	C	X	C	X	X	C	X	X	C	C	C	C	C	C	C	C	X	X	C	C	C	P	P	P
17	M	C	C	X	C	X	X	C	C	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
18	Cxt	C	P	P	C	X	X	P	P	P	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
19	bbp,dddd(q,dnp)	C	P	X	C	X	X	C	X	X	C	C	C	C	C	C	C	C	X	X	C	C	C	P	P	P
20	bp,dddnp,g	C	X	X	X	X	X	C	X	X	C	C	C	C	C	C	C	C	X	X	C	C	C	P	P	P
21	bp,dddp,g	C	X	X	X	X	X	C	X	X	C	C	C	C	C	C	C	C	X	X	C	C	C	C	C	X
22	bp,ddnp,g	C	X	X	X	X	X	C	X	X	C	C	C	C	C	C	C	C	X	X	C	C	C	P	P	P
23	bp,ddp	C	C	X	C	X	X	C	C	X	C	C	C	C	C	C	C	C	C	X	C	C	C	C	C	X
24	bq,d	C	C	X	C	C	X	C	C	X	C	C	C	C	C	C	C	C	C	X	C	C	C	C	C	X
25	bq,ddddddnq	C	P	X	C	X	X	C	X	X	C	C	C	C	C	C	C	C	X	X	C	C	C	P	P	P
26	bq,dddddnq	C	X	X	C	X	X	C	C	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
27	d(bp,ddnq),ddbq	C	P	P	C	X	X	P	P	P	C	C	C	C	C	C	C	C	X	X	C	C	C	P	P	P
28	d(dddddnq,ddddbq)	P	P	P	P	P	X	P	P	X	C	C	C	C	C	C	C	C	X	X	C	C	C	P	P	P
29	d(ddddnq,dddbq)	P	P	P	P	P	X	P	P	P	C	C	C	C	C	C	C	C	X	X	C	C	C	P	P	P
30	d(ddnq,dbq)	P	P	P	P	P	X	P	P	P	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
31	d(dnq,dbq)	C	C	X	C	X	X	C	C	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
32	d(dnq,dddbq)	P	P	P	C	X	X	C	X	X	C	C	C	C	C	C	C	C	X	X	C	C	C	P	P	P
33	dbp,dd(np,bq)	C	P	P	C	X	X	P	P	P	C	C	C	C	C	C	C	C	X	X	C	C	C	P	P	P
34	dbp,dddddnp	C	P	P	C	X	X	C	X	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
35	dbp,ddnp	C	P	P	C	X	X	P	P	P	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
36	dbq,d(dnq,ddp)	C	P	P	C	X	X	P	P	P	C	C	C	C	C	C	C	C	X	X	C	C	C	P	P	P
37	dd(bq,ddddnq)	P	P	P	C	X	X	C	X	X	C	C	C	C	C	C	C	C	X	X	C	C	C	P	P	P
38	dd(dnq,dbq)	C	X	X	C	X	X	C	C	X	C	C	C	C	C	C	C	C	C	X	C	C	C	P	P	P
39	ddddddnq,dp,bbp	C	X	X	C	X	X	C	X	X	C	C	C	C	C	C	C	C	X	X	C	C	C	P	P	P

• • •

Precedence of Predicate Terms

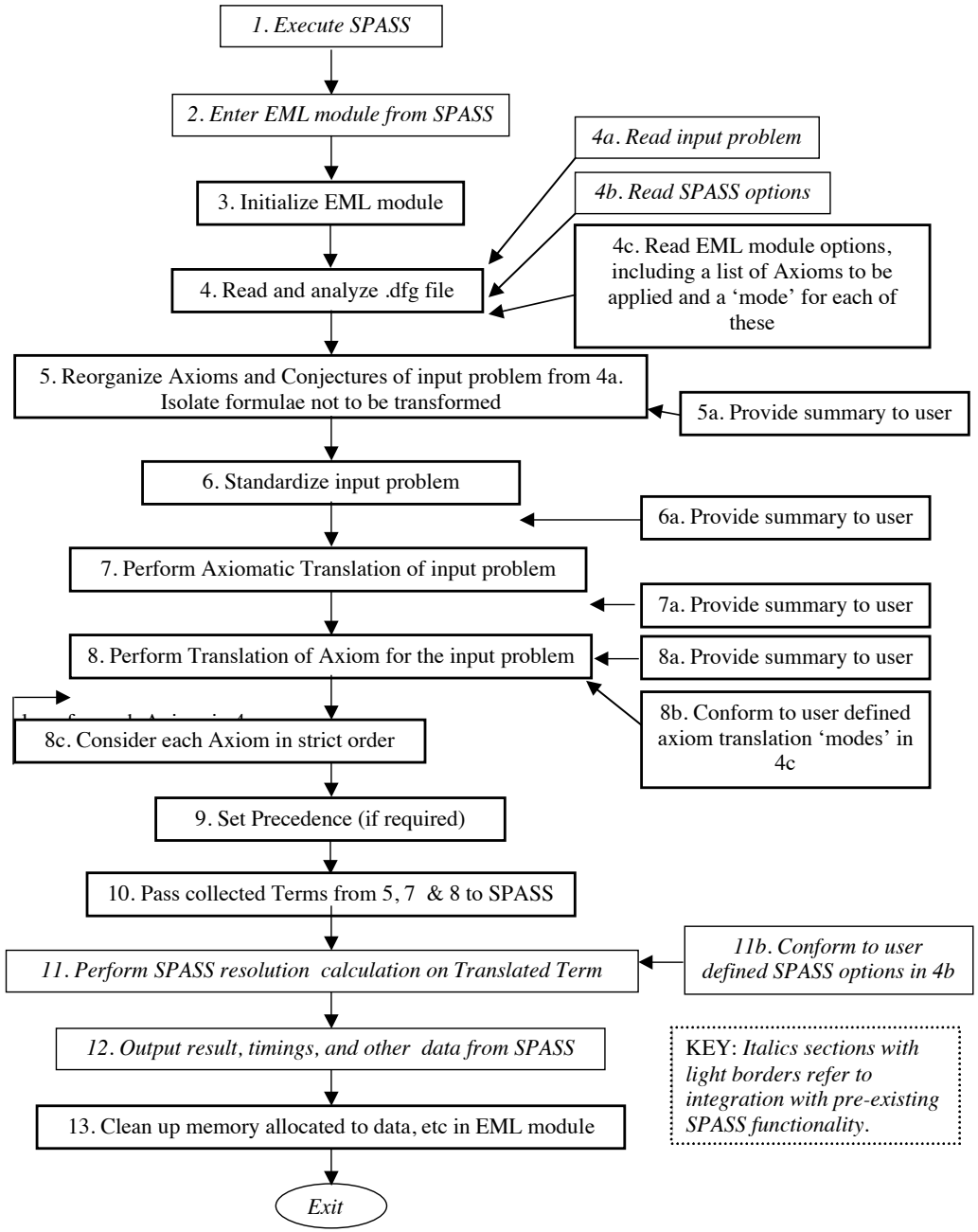
The eml-module automatically sets precedence according to 'Inverse popularity & sub formulae ordering' rules

- Accessibility relations **R >** other predicate terms **Q**
- Ordering of Q–terms is based on frequency that predicate appears in the `FINAL TERM`
  - Least frequently occurring symbol is assigned a higher precedence
- The terms are further ordered (at lower priority) so that
  - Q-terms containing 'and' terms > Q-terms without 'and' terms (later is a sub-term of former)
  - deep Q-terms > shallow Q-terms (so $Q_{\Box\Box\Box p} > Q_{\Box\Box p}$)
  - Qp > Qq, by simple String comparison on p and q (using `strcmp()`).
  - $Q_{\neg Term} > Q_{Term}$ ($\neg p > p$)

```
Re-Organized Precedence = equal > true > false > div > id > R > r > p
> q > Rr > Qnot_and_not_box_r_not_box_r_p_not_box_r_p > Qnot_p > Qp >
Qand_not_box_r_not_box_r_p_not_box_r_p > Qnot_box_r_not_box_r_p >
Qbox_r_not_box_r_p > Qnot_box_r_p > Qbox_r_p
```

- This is important for *decidability*
- Major increases in speed if an optimal precedence is chosen
- Likely to be an area for further investigation

Overview of Sequence of Tasks in SPASS Execution for Axiomatic Translation of An Input Problem.



- Implemented in C (using Europa Eclipse)
- Compiles with gcc; for Mac OSX (linux, Windows/Cygwin, Solaris)
- Design allows same code to be used for uni/multi-modal modal logic and uni/bi-modal axioms
- Suitable for educational & research use
- Test & Requirements, Specification & Implementation documentation provided
- Sufficiently fast

  Over all the test examples, with all the applied axioms and axiom combinations (16971 cases) the execution time for the EML to FOL translation ranges up to a maximum of 1.81 seconds, with the following statistics.

< 0.01 seconds	61 %
0.01-0.1 seconds	38 %
0.1-2 seconds	< 1%.

- No memory leaks detected when executed through Valgrind (on linux)
- Range of other technologies used: Bison, Flex, XHTML, JavaScript, CSS, PERL

The Web based extended-SPASS interface (1) (see http://project.kjsmith.net)

KJSpass

http://www2.cs.man.ac.uk/~smithk/KJSpass/main.html

Apple Tesco QVC Hotmail Webmail cs.KJSmith.net Radio4 BBC 7 YouTube Weather Hazel@kjsmith.net 1&1 Halifax QVC-USA

KJSpass

Main | Parameters | + | Sc | History | Help

Script

Result

```
begin_problem(DL_Example).
list_of_descriptions.
  name({*Renate*}).
  author({*Renate*}).
  status(unsatisfiable).
  description({* A description logic example. *}).
end_of_list.

list_of_symbols.
  functions[
      (adam,0),  (bob,0),  (cauliflower,0),  (cheddar,0)
  ].
  predicates[
      (plant,0),          (Plant,1),          (cheese,0),          (Cheese,1),
      (food,0),           (Food,1),           (person,0),          (Person,1),
      (vegetarian,0),     (Vegetarian,1),     (cheese_lover,0),    (Cheese_lover,1),
      (eat,0),            (Eat,2),            (sibling_of,0),      (Sibling_of,2)
  ].
  translpairs[
      (plant,Plant),              (cheese,Cheese),              (food,Food),
      (vegetarian,Vegetarian),    (cheese_lover,Cheese_lover),
      (eat,Eat),                  (sibling_of,Sibling_of)
  ].
```

More Info.

Clear

Go

Download

20:04:00

Exptal Code

Command Line

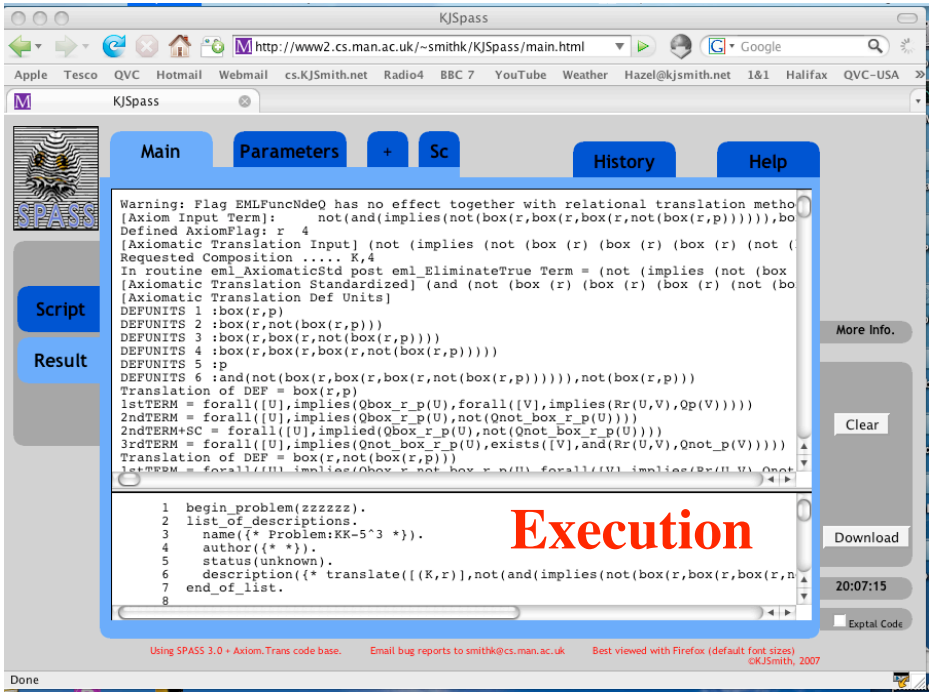
-EMLTranslation=0 -PProblem=1 -PGiven=1 -DocProof=1

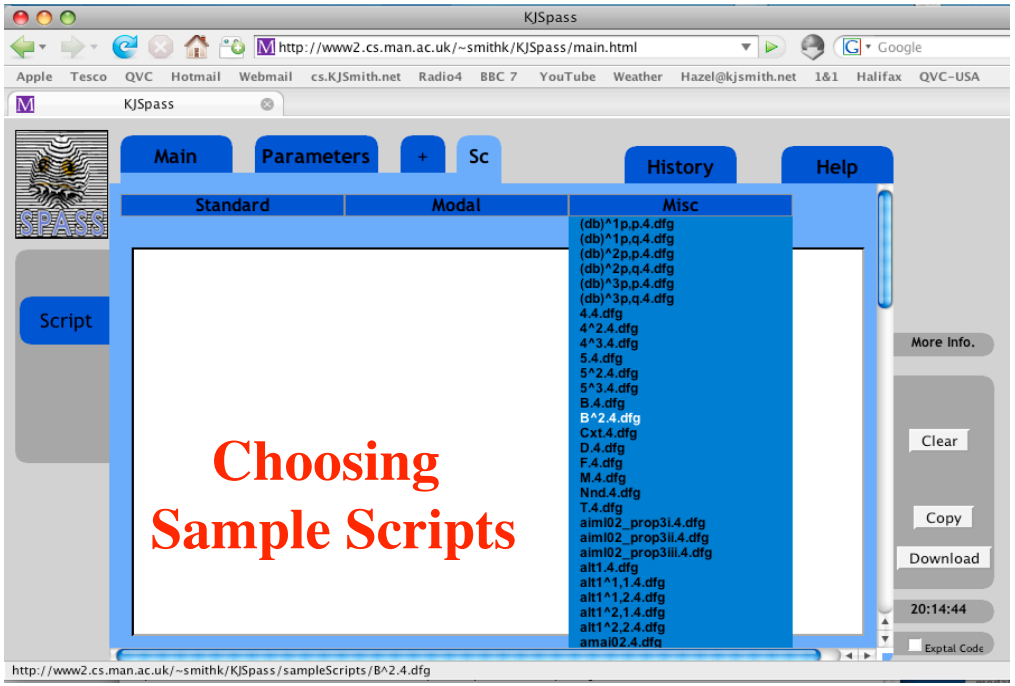
Using SPASS 3.0 + Axiom.Trans code base. Email bug reports to smithk@cs.man.ac.uk Best viewed with Firefox (default font sizes)

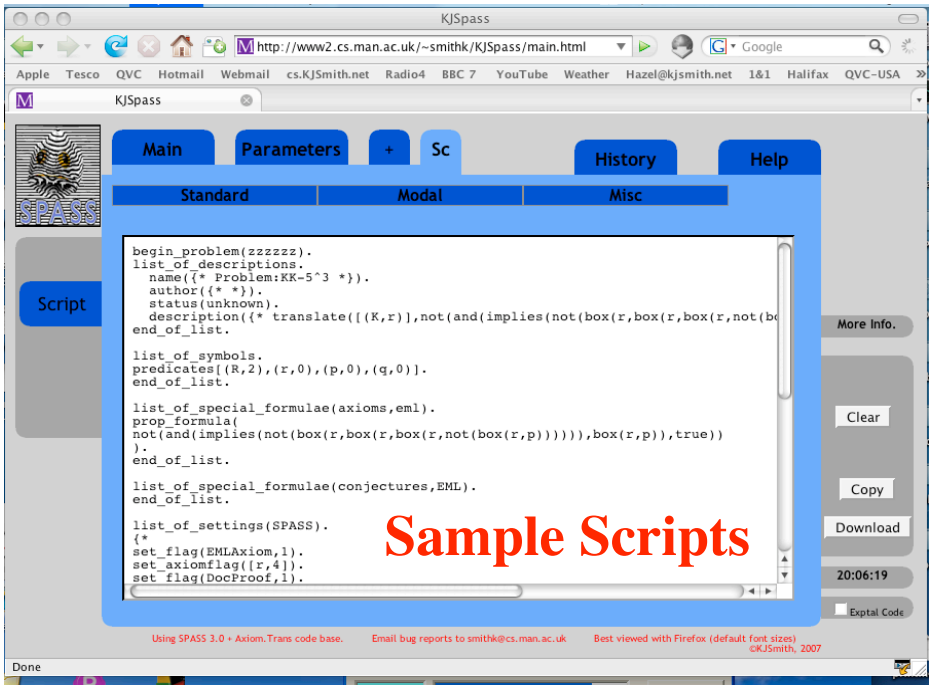
©KJSmith, 2007

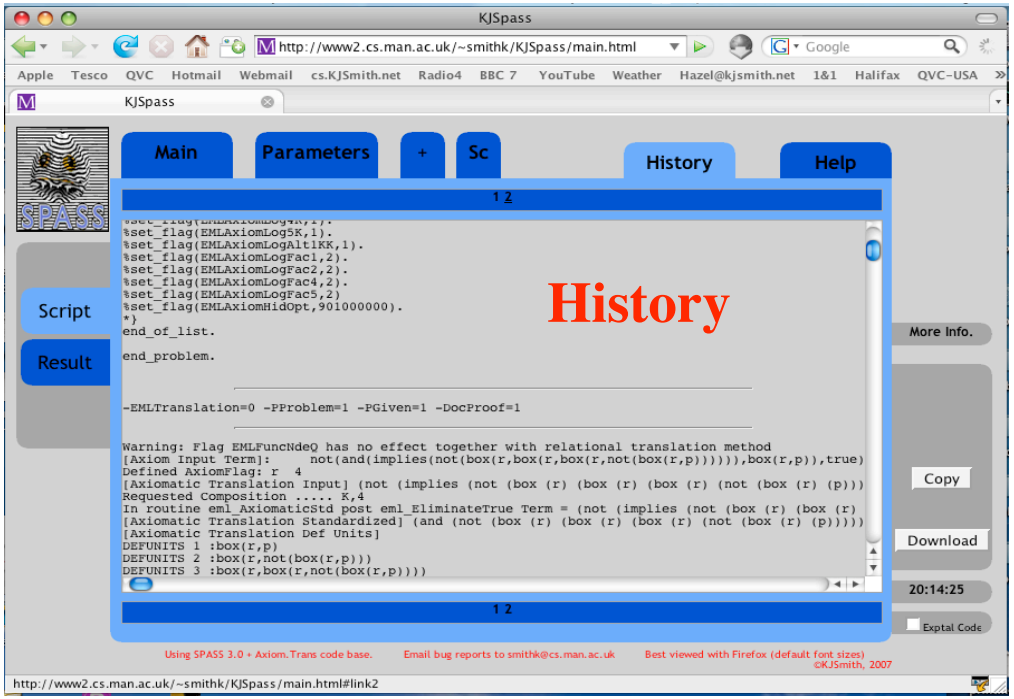
Done

The Web based extended-SPASS interface (2) (see http://project.kjsmith.net)









The Web based extended-SPASS interface (3)

Choose Axioms

axiom T | axiom B | axiom D | axiom 4 | axiom 5 | axiom alt1
axiom 4^2 | 4^3 | 5^2 | 5^3 | alt1^11 | alt1^12 | alt1^21 | alt1^22

Override Protections: Off / On

Set Parameters Axioms

EMLAxiom	0 / 1	EMLClassLog4	0
EMLAxiomLog4	0	EMLClassLog5	0
EMLAxiomLog5	0	EMLClassLogD	0
EMLAxiomLogD	0	EMLClassLogT	0
EMLAxiomLogT	0	EMLClassLogB	0
EMLAxiomLogB	0	EMLClassLogAlt1	0
EMLAxiomLogAlt1	0	EMLClassLog4K	0
EMLAxiomLog4K	0	EMLClassLog5K	0
EMLAxiomLog5K	0	EMLClassLogAlt1KK	0
EMLAxiomLogAlt1KK	0	EMLClassLogFac1	0
EMLAxiomLogFac1	0	EMLClassLogFac2	0
EMLAxiomLogFac2	0	EMLClassLogFac4	0
EMLAxiomLogFac4	0	EMLClassLogFac5	0
EMLAxiomLogFac5	0		

EMLAxiomOptions	0 9	1 0	2 0
	3 0	4 0	5 0
	6 0	7 0	8 0
EMLAxiomHidOpt	0 9	1 0	2 0
	3 0	4 0	5 0
	6 0	7 0	8 0

Do not use composition for axiomatic translation of Axiom B	off	on	
Do not use composition for axiomatic translation of Axiom D	off	on	
Do not use composition for axiomatic translation of Axiom alt1	off	on	
Do not use composition for axiomatic translation of Axiom 4	off	on	
Do not use composition for axiomatic translation of Axiom 5	off	on	
Do not use composition for axiomatic translation of Axiom 4^K	off	on	
Do not use composition for axiomatic translation of Axiom 5^K	off	on	
Do not use composition for axiomatic translation of Axiom alt1^kk	off	on	
Do not use set precedence.	off	on	
Ignore logic flag warnings (activate experimental mode)	off	on	
Exclude optional positive shortcuts in axiomatic translation	off	on	
Do not sort term names	off	on	
Do not use composition for any axiom in axiomatic translation	off	on	
Always Use Substituted Symbol Names	off	on	
Do Not Use Print Statements	off	on	full
NeverUsePositiveShortcuts	off	on	

- Setting extended-SPASS parameters
- Calculator for options flags

- **use it online at http://project.kjsmith.net**
- **download it at http://project.kjsmith.net**
- **additional feature requests**

john@kjsmith.net

Input … $\neg(\Box(r,p)\rightarrow p)$ in $K5_oT$

Standardized … $\Box(r,p)\wedge\neg p$ or $\wedge(\Box(r,p),\neg p)$

Instantiation set … { $\underline{\underline{\Box(r,p)}}$, $\underline{\underline{p}}$, $\underline{\underline{\Box(r,p)\wedge\neg p}}$ }

DEF[$\Box(r,p)$]	$\forall x(Q_{\Box rp}(x) \leftrightarrow \neg Q_{\neg\Box rp}(x))$	K
	$\wedge\ \forall x(Q_{\Box rp}(x) \rightarrow \forall y(R_r(x,y) \rightarrow Q_p(y)))$	
	$\wedge\ \forall x(Q_{\neg\Box rp}(x) \rightarrow \exists y(R_r(x,y) \wedge Q_{\neg p}(y)))$	
DEF[$\wedge(\Box(r,p),\neg(p))$]	$\wedge\ \forall x(Q_{\wedge\Box rp\neg p}(x) \leftrightarrow \neg Q_{\neg\wedge\Box rp\neg p}(x))$	
	$\wedge\ \forall x(Q_{\wedge\Box rp\neg p}(x) \rightarrow (Q_{\neg p}(x) \wedge Q_{\Box rp}(x)))$	
	$\wedge\ \forall x(Q_{\neg\wedge\Box rp\neg p}(x) \rightarrow (Q_p(x) \vee Q_{\neg\Box rp}(x)))$	
DEF[p]	$\wedge\ \forall x(Q_p(x) \leftrightarrow \neg Q_{\neg p}(x))$	
$\Box$ Instantiation set … { $\Box(r,p)$ }		5
5[$\Box(r,p)$]	$\wedge\ \forall x(Q_{\Box rp}(x) \vee \forall y(\neg R_r(x,y) \vee \neg Q_{\Box rp}(y)))$	
5[$\Box(r,\neg(\Box(r,p)))$]	$\wedge\ \forall x(Q_{\Box r\neg\Box rp}(x) \vee \forall y(\neg R_r(x,y) \vee \neg Q_{\Box r\neg\Box rp}(y)))$	5_o
DEF[$\Box(r,\neg(\Box(r,p)))$]	$\wedge\ \forall x(Q_{\Box r\neg\Box rp}(x) \leftrightarrow \neg Q_{\neg\Box r\neg\Box rp}(x))$	
	$\wedge\ \forall x(Q_{\Box r\neg\Box rp}(x) \rightarrow \forall y(R_r(x,y) \rightarrow Q_{\neg\Box rp}(y)))$	
	$\wedge\ \forall x(Q_{\neg\Box r\neg\Box rp}(x) \rightarrow \exists y(R_r(x,y) \wedge Q_{\Box rp}(y)))$	
$\Box$ Instantiation set … { $\Box(r,p)$, $\Box(r,\neg(\Box(r,p)))$}		T
T[$\Box(r,p)$]	$\wedge\ \forall x(\neg Q_{\Box rp}(x) \vee Q_p(x))$	
T[$\Box(r,\neg(\Box(r,p)))$]	$\wedge\ \forall x(\neg Q_{\Box r\neg\Box rp}(x) \vee Q_{\neg\Box rp}(x))$	

```
SPASS -EMLAxiomLog5=1 -EMLAxiomLogT=1 T.dfg

[Axiom Input Term]: not(implies(box(r,p),p))
Axiomatic Translation version 1.0.2 @KJSmith 2008.
[Axiomatic Translation Input] not(implies(box(r,p),p))
Requested Composition ..... K,5o,T
[Axiomatic Translation Standardized] and(box(r,p),not(p))
[Axiomatic Translation Def Units]
DEFUNITS 1 :box(r,p)
DEFUNITS 2 :p
DEFUNITS 3 :and(box(r,p),not(p))
Translation of DEF = box(r,p)
2ndTERM =
forall([U],implies(Qbox_r_p(U),not(Qnot_box_r_p(U))))
2ndTERMSC =
forall([U],implied(Qbox_r_p(U),not(Qnot_box_r_p(U))))
1stTERM =
forall([U],implies(Qbox_r_p(U),forall([V],implies(Rr(U,V),Q
p(V)))))
3rdTERM =
forall([U],implies(Qnot_box_r_p(U),exists([V],and(Rr(U,V),Q
not_p(V)))))
Translation of DEF = and(box(r,p),not(p))
2ndTERM =
forall([U],implies(Qand_box_r_p_not_p(U),not(Qnot_and_box_r
_p_not_p(U))))
2ndTERMSC =
forall([U],implied(Qand_box_r_p_not_p(U),not(Qnot_and_box_r
_p_not_p(U))))
1stTERM =
forall([U],implies(Qand_box_r_p_not_p(U),and(Qnot_p(U),Qbox
_r_p(U))))
3rdTERM =
forall([U],implies(Qnot_and_box_r_p_not_p(U),or(Qp(U),Qnot_
box_r_p(U))))
Translation of DEF = p
2ndTERM = forall([U],implies(Qp(U),not(Qnot_p(U))))
2ndTERMSC = forall([U],implied(Qp(U),not(Qnot_p(U))))
[Axiomatic Translation: Logic K]
and(exists([U],Qand_box_r_p_not_p(U)),forall([U],implies(Qn
ot_box_r_p(U),exists([V],and(Rr(U,V),Qnot_p(V))))),forall([
U],implies(Qbox_r_p(U),forall([V],implies(Rr(U,V),Qp(V)))))
,forall([U],implied(Qbox_r_p(U),not(Qnot_box_r_p(U)))),fora
ll([U],implies(Qbox_r_p(U),not(Qnot_box_r_p(U)))),forall([U
],implies(Qnot_and_box_r_p_not_p(U),or(Qp(U),Qnot_box_r_p(U
)))),forall([U],implies(Qand_box_r_p_not_p(U),and(Qnot_p(U)
,Qbox_r_p(U)))),forall([U],implied(Qand_box_r_p_not_p(U),no
t(Qnot_and_box_r_p_not_p(U)))),forall([U],implies(Qand_box_
[Axiom T]:[r in term:box(r,p)].
forall([U],or(not(Qbox_r_p(U)),Qp(U)))
[Axiom T]:[r in term:box(r,not(box(r,p)))].
forall([U],or(not(Qbox_r_not_box_r_p(U)),Qnot_box_r_p(U)))
 0: Qand_box_r_p_not_p
 1: Qbox_r_p
 2: Qnot_box_r_p
 3: Qp
 4: Rr
 5: Qnot_p
 6: Qnot_and_box_r_p_not_p
 7: Qbox_r_not_box_r_p
 8: Qnot_box_r_not_box_r_p
Using Inverse Popularity + Subformulae Ordering.
Re-Organized Precedence = equal > true > false > div > id >
R > r > p > q > Rr > Qnot_and_box_r_p_not_p >
Qnot_box_r_not_box_r_p > Qand_box_r_p_not_p > Qnot_p > Qp >
Qbox_r_not_box_r_p > Qnot_box_r_p > Qbox_r_p
FINAL TERM:
     and(exists([U],Qand_box_r_p_not_p(U)),forall([U],impli
es(Qnot_box_r_p(U),exists([V],and(Rr(U,V),Qnot_p(V))))),for
all([U],implies(Qbox_r_p(U),forall([V],implies(Rr(U,V),Qp(V
)))))),forall([U],implied(Qbox_r_p(U),not(Qnot_box_r_p(U))))
,forall([U],implies(Qbox_r_p(U),not(Qnot_box_r_p(U)))),fora
ll([U],implies(Qnot_and_box_r_p_not_p(U),or(Qp(U),Qnot_box_
r_p(U)))),forall([U],implies(Qand_box_r_p_not_p(U),and(Qnot
_p(U),Qbox_r_p(U)))),forall([U],implied(Qand_box_r_p_not_p(
U),not(Qnot_and_box_r_p_not_p(U)))),forall([U],implies(Qand
_box_r_p_not_p(U),not(Qnot_and_box_r_p_not_p(U)))),forall([
U],implied(Qp(U),not(Qnot_p(U)))),forall([U],implies(Qp(U),
not(Qnot_p(U)))),and(forall([U],implies(Qnot_box_r_not_box_
r_p(U),exists([V],and(Rr(U,V),Qbox_r_p(V))))),forall([U],im
plies(Qbox_r_not_box_r_p(U),forall([V],implies(Rr(U,V),Qnot
_box_r_p(V))))),forall([U],implied(Qbox_r_not_box_r_p(U),no
t(Qnot_box_r_not_box_r_p(U)))),forall([U],implies(Qbox_r_no
t_box_r_p(U),not(Qnot_box_r_not_box_r_p(U))))),forall([U],o
r(Qbox_r_not_box_r_p(U),forall([V],or(not(Rr(U,V)),not(Qbox
_r_not_box_r_p(V))))),forall([U],or(Qbox_r_p(U),forall([V]
,or(not(Rr(U,V)),not(Qbox_r_p(V))))),forall([U],or(not(Qbo
x_r_not_box_r_p(U)),Qnot_box_r_p(U))),forall([U],or(not(Qbo
x_r_p(U)),Qp(U))))

SPASS V 3.0
SPASS beiseite: Proof found.
Problem: T.dfg
SPASS derived 7 clauses, backtracked 0 clauses and kept 27
clauses.
SPASS allocated 574 KBytes.
```